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ERROR PROBABILITIES FOR MAXIMUM LIKELIHOOD DETECTION OF M-ary POISSON PROCESSES IN POISSON NOISE

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SUMMARY

In this problem we have considered some of the recent results in the detection of a Poisson-distributed signal in Poisson noise. Curves for error probabilities are presented for the case of detecting one of M equiprobable signals over a broad range of parameter values. Implicit in these results for system applications is the use of "photon counting" receivers. Attention is given to the optical communication and radar problems for this receiver structure and significant parameters are translated into those used in the report. A complete description of the computational procedures used for making the error probability calculations is given.

INTRODUCTION

The purpose of this paper is to summarize some of the recent results (refs. 1-5) concerning the maximum likelihood detection of Poisson-distributed signals in the presence of Poisson-distributed noise and to tabulate the resultant error probabilities over a broad range of signal and noise when optimum signal design for maximum distance is used. Furthermore, the report shows how these results can be applied to the direct detection of optical signals, with the optimum detector being a counter of photoelectrons. This form of detector can be implemented in the visible portion of the spectrum where photomultipliers exist.

The tabulation is presented in two forms. The first form is related to the detection of M-ary signals and is applicable to the communications problem. The second form is related to the range bin problem in pulsed radar systems. It is felt that the values of the parameters will be applicable to most problems of this type.

BACKGROUND

When a classically describable field is incident upon a photodetecting surface, the probability dp of releasing a photoelectron in an interval dt over a surface area $d\sigma$ is (ref 6):

$$dp = \alpha I\ (t, \sigma)\ dt\ d\sigma$$

where $I\ (t,\sigma)$ is the intensity of the field and α is equal to η/hf in which η is the quantum efficiency, h is Planck's constant, and f is the frequency. For a surface of area A:

$$\alpha\ dt \int_A d\sigma\ I\ (t,\sigma) = \alpha P\ (t)\ dt$$

where P (t) is the total collected power at time t. Suppose that the power is related to the incident particle rate n (t) by

$$P\ (t) = n\ (t)\ hf.$$

The probability of releasing a photoelectron in a time dt is then

$$\alpha P\ (t)\ dt = \eta n\ (t)\ dt.$$

With this assumption, the probability of releasing K photoelectrons p(K) in a finite interval ΔT is (ref. 7):

$$p(K) = \frac{\left[\int_t^{t + \Delta t} \eta n(t)\ dt \right]^K}{K!} \exp \left[- \int_t^{t + \Delta t} \eta n(t)\ dt \right]$$

A system of events obeying this density distribution is called a Poisson process (ref. 8). If the system of events is stationary, we can replace the time average by an ensemble average, as

$$p(K) = \frac{(\eta \bar{n}\ \Delta T)^K}{K!} \exp \left[-\eta \bar{n}\ \Delta T \right].$$

The system of events would then be called a stationary Poisson process (ref. 8).

Strictly speaking, p (K) is a conditional density and should be written as $p\left[K/\overline{n}(t)\right]$ since, in general, n(t) itself is a sample from a random process. However, in communications, one is interested in "designing" the waveform n(t) to satisfy certain desirable features. Consequently, n(t) is assumed to be deterministic.

Let us assume that we are monitoring the current output of a unity quantum efficiency photodetector in an interval $(0, T)$ and can distinguish all events. Let us further assume that one of two different rates was sent, resulting in one of two received $n_1(t)$, $n_2(t)$, plus a stationary additive constant rate $\bar{n}_n$ corresponding to noise. We wish to formulate the maximum likelihood detection procedure for determining which rate is imbedded in the received signal. If the possible transmitted rates are both band-limited to B, then so are $n_1(t)$ and $n_2(t)$. We can therefore partition the $(0, T)$ interval into M subintervals $t_{i+1} - t_i = \Delta T$ (ref. 1) where

$$0 = t_o < t_1 < \dots . < t_{M-1} < t_M = T,$$

with $\Delta T \leq \frac{1}{2B}$, and consider the quantized version of the possible rates. That is:

$$n_{ji} = \frac{1}{\Delta T} \int_{t_i}^{t_{i+1}} n_j \, (t)\, dt; \qquad (t_i \leq t \leq t_{i+1}),$$

$$j = 1, 2.$$

To accomplish the detection we consider two hypotheses: H_1 -- rate $n_1(t)$ and noise $\bar{n}_n$ are present; H_2 -- rate $n_2(t)$ and noise $\bar{n}_n$ are present. We now consider a vector space of M dimensions where each dimension represents the number of events K_i observed in the corresponding interval. Since the number of events is independent from interval to interval, the vector $K = (K_1 \dots\dots, K_M)$ has a conditional probability given H_1 of

$$p(\underline{K} \mid H_1) = \prod_{i=1}^{M} \frac{[(n_{1i} + \bar{n}_n)\Delta T]^{K_i}}{K_i!} e^{-(n_{1i} + \bar{n}_n)\Delta T}$$

and a conditional probability, given H_2, of

$$p(\underline{K} | H_2) = \prod_{i=1}^{M} \frac{\left[(n_{2i} + \overline{n}_n)\Delta T \right]^{K_i}}{K_i!} e^{-(n_{2i} + \overline{n}_n)\Delta T}$$

The likelihood ratio Λ is then defined as

$$\Lambda(K) = \frac{p(\underline{K} | H_1)}{p(\underline{K} | H_2)} = \prod_{i=1}^{M} \left[\frac{(n_{1i} + \overline{n}_n)}{(n_{2i} + \overline{n}_n)} \right]^{K_i} e^{-(n_{1i} - n_{2i})\Delta T}$$

and the maximum likelihood detection criterion requires, after observing $\underline{K}$, a comparison of $\Lambda(\underline{K})$ to a threshold c. If $\Lambda \geq c$, we decide rate $n_1(t)$ is imbedded within the received process, and if $\Lambda < c$, we decide rate $n_2(t)$. Since the log function is monotonic, we can also make a decision based upon

$$\log \Lambda \lesseqgtr \log c$$

where $\lesseqgtr$ denotes the threshold comparison.

Thus $\log \Lambda$ is

$$\sum_{i=1}^{M} K_i \log \left[\frac{n_{1i} + \overline{n}_n}{n_{2i} + \overline{n}_n} \right] - \sum_{i=1}^{M} (n_{1i} - n_{2i}) \Delta T \gtreqless \log c,$$

which can be rewritten as

$$\Lambda' \lesseqgtr \left[\log c + \sum_{i=1}^{M} (n_{1i} - n_{2i}) \Delta T \right] \tag{1}$$

where

$$\Lambda' = \sum_{i=1}^{M} K_i \log \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) - \sum_{i=1}^{M} K_i \log \left(1 + \frac{n_{2i}}{\bar{n}_n}\right) . \tag{2}$$

Thus the maximum likelihood detection test involves, equivalently, a comparison of the quantity Λ' in Eq. (2) to the new threshold in Eq. (1).

"Distance" Considerations

Let us first consider the case where $n_{2i} = 0$ for all i and we are detecting the rate $n_i(t) + n_n$ versus $\bar{n}_n$ alone (i.e., signal to no signal). Then the test in Eq. (1) becomes

$$\Lambda' = \sum_{i=1}^{M} K_i \log \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) \gtrless \log c + \sum_{i=1}^{M} n_{1i} \Delta T$$

Clearly, since K_i is Poisson-distributed:

$$E_{\underline{K}} [\Lambda'/H_1] = \sum_{i=1}^{M} (n_{1i} + \bar{n}_n) \log \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) \Delta T$$

and

$$E_{\underline{K}} [\Lambda'/H_2] = \sum_{i=1}^{M} \bar{n}_n \log \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) \Delta T .$$

We define the "distance" D between the two hypotheses as

$$\begin{aligned} D &= E_{\underline{K}} [\Lambda'/H_1] - E_{\underline{K}} [\Lambda'/H_2] \\ &= \sum_{i=1}^{M} n_{1i} \log \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) \Delta T . \end{aligned}$$

We would like to investigate maximum and minimum of the distance

$$D = \sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] \Delta T$$

subject to the energy constraint

$$\sum_{i=1}^{M} n_{1i} = K' = \frac{K_s}{\Delta T} \quad .$$

Using a Lagrange multiplier we seek the minimum of

$$I = \Delta T \sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] + \lambda \sum_{i=1}^{M} n_{1i}$$

which requires

$$\frac{\partial I}{\partial n_{1i}} = \Delta T \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] + \frac{\Delta T \, n_{1i}}{n_{1i} + \bar{n}_n} + \lambda = 0$$

and is a set of M equations that must be satisfied for all i . The solution is $n_{1i} = n$ for all i yielding

$$D_{min} = \sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] \Delta T$$

$$= \Delta T \, Mn \log \left[1 + \frac{n}{\bar{n}_n} \right] = K_s \log \left[1 + \frac{K_s}{T\bar{n}_n} \right]$$

which can be identified as a minimum.

The maximum value can be obtained by noting that

$$\sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] \le \left[\sum_{i=1}^{M} (n_{1i})^2 \right]^{1/2} \left[\sum_{i=1}^{M} \log^2 \left(1 + \frac{n_{1i}}{\bar{n}_n}\right) \right]^{1/2}$$

$$\le \left[\sum_{i=1}^{M} n_{1i} \right] \left[\sum_{i=1}^{M} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] \right]$$

with the equality occurring for $n_{1i} = K' \delta_{ij}$.

The maximum value for the sum becomes:

$$D_{max} = \sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] \Delta T = K_s' \log \left[1 + \frac{K_s}{\bar{n}_n \Delta T} \right]$$

Thus, the distance D is maximized when the rate $n_1(t)$ is concentrated in one ΔT interval. Hence, the signal maximizing detectability* (refs. 1, 2) also maximizes distance. Notice also, that for the important case when $\bar{n}_n \gg n_{1i}$, for all i ,

$$D = \frac{\sum_{i=1}^{M} n_{1i}^2 \Delta T}{\bar{n}_n}$$

with distance and detectability identical.

The maximization can also be obtained for the case $n_{2i} \neq 0$.

$$D = E_{\Lambda'}[\Lambda'/H_1] - E_{\Lambda'}[\Lambda'/H_2] = \sum_{i=1}^{M} (n_{1i} - n_{2i}) \log \left(\frac{n_{1i} + \bar{n}_n}{n_{2i} + \bar{n}_n} \right) \Delta T$$

$$= \sum_{i=1}^{M} \log \left[\frac{n_{1i} + \bar{n}_n}{n_{2i} + \bar{n}_n} \right]^{(n_{1i} - n_{2i})} \Delta T .$$

Since n_{1i} and n_{2i} must be non-negative, the log term is maximized for all i by having $n_{2i} = 0$ when n_{1i} is not, and vice versa. This maximizes

$$\left[\frac{n_{1i} + \bar{n}_n}{n_{2i} + \bar{n}_n} \right]^{(n_{1i} - n_{2i})}$$

by giving the largest magnitude to both the bracketed term and the exponent simultaneously and leaves the sign of the log positive.

* Signal-to-noise ratio

Therefore, the two signals should be disjoint in time. On the assumption that they are disjoint, then:

$$D = \sum_{i=1}^{M} n_{1i} \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] + \sum_{i=1}^{M} n_{2i} \log \left[1 + \frac{n_{2i}}{\bar{n}_n} \right] .$$

To maximize D, we have only to notice that this is identical to maximizing each signal independently or to concentrate each signal in a different time interval. The optimum processor, therefore, calculates

$$\Lambda' = \sum_{i=1}^{M} K_i \log \left[1 + \frac{n_{1i}}{\bar{n}_n} \right] - \sum_{i=1}^{M} K_i \log \left[1 + \frac{n_{2i}}{\bar{n}_n} \right]$$

which is compared to the threshold

$$\log c + \sum_{i=1}^{M} (n_{1i} - n_{2i}) \Delta T .$$

If the two signals are equiprobable with equal energy

$$\sum_{i=1}^{M} n_{1i} \Delta T = \sum_{i=1}^{M} n_{2i} \Delta T ,$$

c = 1 and the threshold is zero. The H_1, H_2 log likelihood functions are calculated for each choice of waveform $n_i(t)$ and the hypothesis is selected according to the largest result.

If M waveforms are used, the optimum processor would calculate the likelihood function for each of the M choices and select the maximum. Expressed mathematically, one obtains for optimum signal design under an equiprobable, equal energy assumption

$$\Lambda' = K_i \log \left[1 + \frac{K_s}{\bar{n}_n \Delta T} \right] - K_j \log \left[1 + \frac{K_s}{\bar{n}_n \Delta T} \right] \lessgtr 0$$

or $K_i \gtrless K_j$.

Hence, the problem is reduced to counting the number of photoelectrons in each ΔT interval and selecting the interval with the largest count. The probability of correct detection P_D is then:

$$P_D = [\text{Probability that } K_j > K_i \text{ for all } i/n_j \text{ is the transmitted waveform}].$$

$$+ \sum_{r=2}^{M} \frac{1}{r} [\text{Probability that } K_j = K_i \text{ for } r-1 \text{ intervals}/n_j \text{ is the transmitted waveform}].$$

This is then averaged over all choices of waveform, which for the equiprobable set is just P_D .

This can be written

$$P_D = \sum_{x=1}^{\infty} \left\{ \frac{(K_s + \bar{n}_n \Delta T)^x}{(x)\,!} e^{-(K_s + \bar{n}_n \Delta T)} \left[\sum_{i=o}^{x-1} \frac{(\bar{n}_n \Delta T)^i}{i\,!} e^{(\bar{n}_n \Delta T)} \right]^{M-1} \left[\frac{(1+B)^M - 1}{MB} \right] \right\} + \frac{1}{M} e^{-(K_s + \bar{n}_n M \Delta T)} \qquad (3)$$

where

$$B = \frac{(\bar{n}_n \Delta T)^x}{x\,! \sum_{i=o}^{x-1} \frac{(\bar{n}_n \Delta T)^i}{i\,!}}$$

The error probability $P_E = 1 - P_D$.

Presentation of the Data

The data are presented in two forms. In the first form (Program 1, Figures 1-10) P_E is plotted versus M for various values of K_s and $K_n = \bar{n}_n \Delta T$. These parameters are directly related to the received signal and noise energies, respectively. Thus, if one determines from the range equation that P_s is the received signal peak power, then

$$K_s = \frac{\eta P_s \Delta T}{hf}$$

whereas if the received average noise power is P_n,

$$K_n = \frac{\eta P_n \Delta T}{hf} \quad . \tag{4}$$

If non-diffraction-limited collecting optics is used, P_n is calculated as

$$P_N = A_R T_o \Omega_r N_\lambda \Delta\lambda \tag{5}$$

with

N_λ = spectral radiance (power/unit area, solid-angle bandwidth)

T_o = fraction of optical transmission through all elements

A_R = area of collector

Ω_r = resolution of receiving collector (solid angle)

$\Delta\lambda$ = optical bandwidth.

In the visible region of the spectrum 5-6000 Å, K_n for non-diffraction-limited optics with a blue sky background can be written as

$$K_n = 8\eta\Delta T(\alpha D)^2 T_o (\Delta\lambda)$$

PROGRAM I

```
   04/30/68                          80/80 LIST
HURW1,C76501,T500,CM50000.   NASA   I01074   KARP3    HURWITZ
 REQUEST TAPE7,LO.     PLOT SAO 568 RING IN.
RFL,50000.
RUN(S,,,,,,77777)
LGO.

      PROGRAM KARP3 (INPUT,OUTPUT,TAPE7,TAPE5=INPUT,TAPE6=OUTPUT)
      DOUBLE PRECISION A,B,EKSN,EL10,ELM,EM,EXK,PD,PE,PI,PX,
     1         S,SAVE,SUM,SUMPI,T,TERM,TERM2,TERM3,TEST,TOTAL,XI,
     2         XK,XKN,XKNMAX,XKNMIN,XKS,XKSN,XM,XM1,XMB,XMBMIN,XX
      DATA IN,NOUT,IPLT,TEST/5,6,7,1.D-24/
      DATA XAX,YAX/8.,11./
      DATA IXMAX,XKNMAX,XKNMIN,XMBMIN/1000,3.D+2,1.D-30,1.D-12/
CCC   KARP/HURWITZ
CCC   A=ABS.VALUE OF DIFFERENCE BETWEEN TWO CONSECUTIVE TERMS
CCC   B DESCRIBED IN WRITE UP
CCC   EKSN=EXP(-XKSN)
CCC   EL10=CONSTANT USED FOR TESTING MAGNITUDES OF NUMBERICAL VALUES
CCC   ELM=LOG(M)
CCC   EM=INITIAL VALUE OF M
CCC   EX=ARRAY FOR LOG(M) FOR EACH KS AND EACH K
CCC   EXK=INPUT OF K
CCC   I=INDEX
CCC   ICUTOF USED TO INSURE NO PREMATURE CUT-OFF OF THE CALCULATION
CCC   IEND=TEST VALUE FOR STOPPING RUN
CCC   IEND MUST BE NINE FOR ALL BUT LAST DATA CARD
CCC   IXMAX=MAX VALUE FOR INDEX IX
CCC   I1 COUNTS NUMBER TOTAL NUMBERS OF PE'S
CCC   J=INDEX
CCC   JMP MUST BE ZERO FOR FIRST DATA CARD
CCC   JMP=TEST VALUE FOR READING VALUES OF K
CCC   NM=NUMBER OF VALUES OF M
CCC   NK=NUMBER OF VALUES OF K
CCC   NN=NPT(I)
CCC   NPT(J) COUNTS NUMBER OF M'S FOR EACH K
CCC   N1=NN+1
CCC   N2=NN+2
CCC   PD DESCRIBED IN WRITE UP
CCC   PE=1-PD
CCC   PI=(KN**I)*EKSN/(I FACTORIAL)
CCC   PX=XKSN*EKSN
CCC   S USED TO TEST MAGNITUDES
CCC   SAVE LAST VALUE OF TERM
CCC   SUM=SUMMATION (KN**I/I)
CCC   SUMPI=SUMMATION (PI)
CCC   T=DUMMY
CCC   TERM=PX*TERM2*TERM3
CCC   TERM2=SUMPI**(M-1)
CCC   TERM3=(1+B)**M-1/(M*B)
CCC   TOTAL=SUMMATION(TERM)
CCC   WY=ARRAY FOR LOG(PE) FOR EACH KS AND EACH K
CCC   X=ARRAY FOR LOG(M) FOR ALL CURVES FOR EACH KS
CCC   XK=EXK(J)
CCC   XKN=KN
CCC   XKNMAX=MAX VALUE OF KN
CCC   XKNMIN=MIN VALUE OF KN
CCC   XKS=KS
```

```
CCC    XKSN=KS+KN
CCC    XM=EM**I
CCC    XMB=XM*B
CCC    XMBMIN=TEST VALUE FOR XMB
CCC    XM1=XM-1
CCC    XX=IX=INDEX OF OUTER LOOP
CCC    Y=ARRAY FOR LOG(PE) FOR ALL CURVES FOR EACH KS
       DIMENSION X(3000),Y(3000),EX(100),WY(100),NPT(100),EXK(100)
       EL10=300.D0*DLOG(10.D0)
       CALL INITPLT(IPLT)
     1 READ (IN,3) EM,XKS, NM,NK,IEND,JMP
       L=0
       IF  (JMP .NE.0) GO TO 2
       READ (IN,4) (EXK(I), I=1,NK)
     4 FORMAT (D10.3)
CCC    LOOP FOR K
     2 DO 70 J=1,NK
       NPT(J)=0
     3 FORMAT (2D10.3,4I5)
       XKN=EXK(J)
CCC   LOOP FOR M
    10 DO 65  I=1,NM
CCC    PRINT HEADINGS FIRST TIME THROUGH THIS LOOP
       IF (I.NE.1) GO TO 14
       WRITE(NOUT,12)XKS,XKN
    12 FORMAT (1H1,40X,3HKS=,D9.2,10X,3HKN=,D9.2//)
       WRITE (NOUT,13)
    13 FORMAT(17X,1HM,20X,2HPD,20X,2HPE,20X,6HLOG(M),16X,7HLOG(PE)//)
    14 XM=EM**I
       XM1=XM-1.D0
CCC    TEST TO INSURE EXP(KN) DOES NOT EXCEED MACHINE CAPACITY
CCC    IF KN GT XKNMAX, INCREASE M AND CONTINUE
       IF (XKN.GT.XKNMAX) GO TO 65
       XKSN=XKS+XKN
CCC    TEST VALUE TO INSURE NO PREMATURE CUTOFF
       ICUTOF=2.D0*XKSN
       EKSN=DEXP(-XKSN)
CCC    FORMATION OF PX FOR X=1
       PX=XKSN*EKSN
       SUM=1.D0
       T=1.D0
CCC    FORMATION OF B FOR X=1
       B=XKN
       XMB=XM*B
CCC    TEST FOR (1+B)**M TOO LARGE
       S=DLOG(1.D0+3)
       S=EL10/XM-S
       IF (S.GT.0.D0) GO TO 17
CCC    VALUE FOR (1+B)**M TOO LARGE FOR MACHINE
       TERM3=0.D0
       GO TO 16
CCC    VALUE FOR (1+B)**M SUFFIENTLY SMALL
    17 TERM3=((1.D0+B)**XM-1.D0)/XMB
CCC    TEST FOR KN NEAR ZERO
    15 IF (XKN.GE.XKNMIN) GO TO 16
       TERM2=1.D0
       TERM3=1.D0
```

```
      GO TO 20
CCC   FORMATION OF PI,SUMPI,TERM2 FOR X=1
   16 PI=DEXP(-XKN)
      SUMPI=PI
      TERM2=PI**XM1
CCC   LOOP FOR SUMMATION OF TERM
   20 TERM=PX*TERM2*TERM3
      TOTAL=TERM
      DO 50 IX=2,IXMAX
      SAVE=TERM
      XX=IX
      X1=XX-1.D0
CCC   FORMATION OF PX FOR X.GE.2
      PX=PX*XKSN/XX
CCC   TEST FOR KN NEAR ZERO
      IF (XKN.LT.XKNMIN) GO TO 45
CCC   FORMATION OF PI FOR X.GE.2
      PI=PI*XKN/X1
CCC   FORMATION OF TERM2 FOR X.GE.2
CCC   FORMATION OF B FOR X.GE.2
      SUMPI=SUMPI+PI
   25 TERM2=(SUMPI)**XM1
      T=T*XKN/X1
      A=SUM
      SUM=SUM+T
      B=XKN*B*A/(XX*SUM)
CCC   TEST FOR (1+B)**M TOO LARGE
      S=DLOG(1.D0+B)
      S=EL10/XM-S
      IF (S.LT.0.D0) GO TO 50
CCC   FORMATION OF TERM3
CCC   TEST FOR XMB LE XMBMIN
CCC   IF MB LE XMBMIN APPROX TERM3 WITH FIRST TWO TERMS OF MB ONLY
   35 XMB=XM*B
      IF (XMB.GT.XMBMIN) GO TO 40
      TERM3=     1.D0+(XM-1.D0)*B/2.D0
      GO TO 45
   40 TERM3=((1.D0+B)**XM-1.D0)/XMB
CCC   FORMATION OF TERM FOR X.GE.2
   45 TERM=PX*TERM2*TERM3
CCC   SUMMATION OF TERMS
      TOTAL=TOTAL+TERM
CCC   TEST IF DIFFERENCE OF TERMS FOR X=N AND X=N+1 IS SUFF. SMALL
      A=DABS(TERM-SAVE)
      IF (A.LT.TEST.AND.IX.GT.ICUTOF) GO TO 55
CCC   TEST TO INSURE AGAINST PREMATURE CUTOFF
   50 CONTINUE
CCC   COMPUTATION OF FINAL VALUE FOR A GIVEN M
   55 PD=TOTAL+DEXP(-(XKS+XM*XKN))/XM
      PE=1.D0-PD
      L=L+1
      ELM=DLOG10(XM)
      X(L)=SNGL(ELM)
      ELM=DLOG10(PE)
      Y(L)=SNGL(ELM)
      NPT(J)=NPT(J)+1
      WRITE (NOUT,60) XM,PD,PE,X(L),Y(L)
```

```
   60 FORMAT(9X,3(D16.9,6X),2(F16.9,6X))
   65 CONTINUE
   70 CONTINUE
      L1=L+1
      L2=L+2
      X(L1)=0.
      X(L2)=0.
      Y(L1)=0.
      Y(L2)=0.
CCC   PLOTTING ROUTINES FOR CAL-COMP PLOTTER
      CALL PLOT(0.,-31.,-3)
      CALL PLOT(0.,2.,-3)
      CALL SCALE (X,XAX,L,1)
      CALL SCALE (Y,YAX,L,1)
      CALL AXIS (0.,0.,6HLOG(M),-6,XAX,0.,X(L1),X(L2))
      CALL AXIS (0.,0,7HLOG(PE),7,YAX,90.,Y(L1),Y(L2))
      I1=0
      DO 90  I=1,NK
      NN=NPT(I)
      DO 85  J=1,NN
      I1=I1+1
      EX(J)=X(I1)
   85 WY(J)=Y(I1)
      N1=NN+1
      N2=NN+2
      EX(N1)=X(L1)
      EX(N2)=X(L2)
      WY(N1)=Y(L1)
      WY(N2)=Y(L2)
      CALL LINE (EX,WY,NN,1,0,0)
   90 CONTINUE
      CALL SYMBOL(0.5,16.0,.3,3HKS=,0.,3)
      CALL NUMBER(1.5,16.0,.3,XKS,0.,-1)
      CALL PLOT(20.,0.,-3)
  110 CONTINUE
      IF(IEND.EQ.9) GO TO 1
      CALL FIN(IPLT)
  120 STOP
      END
```

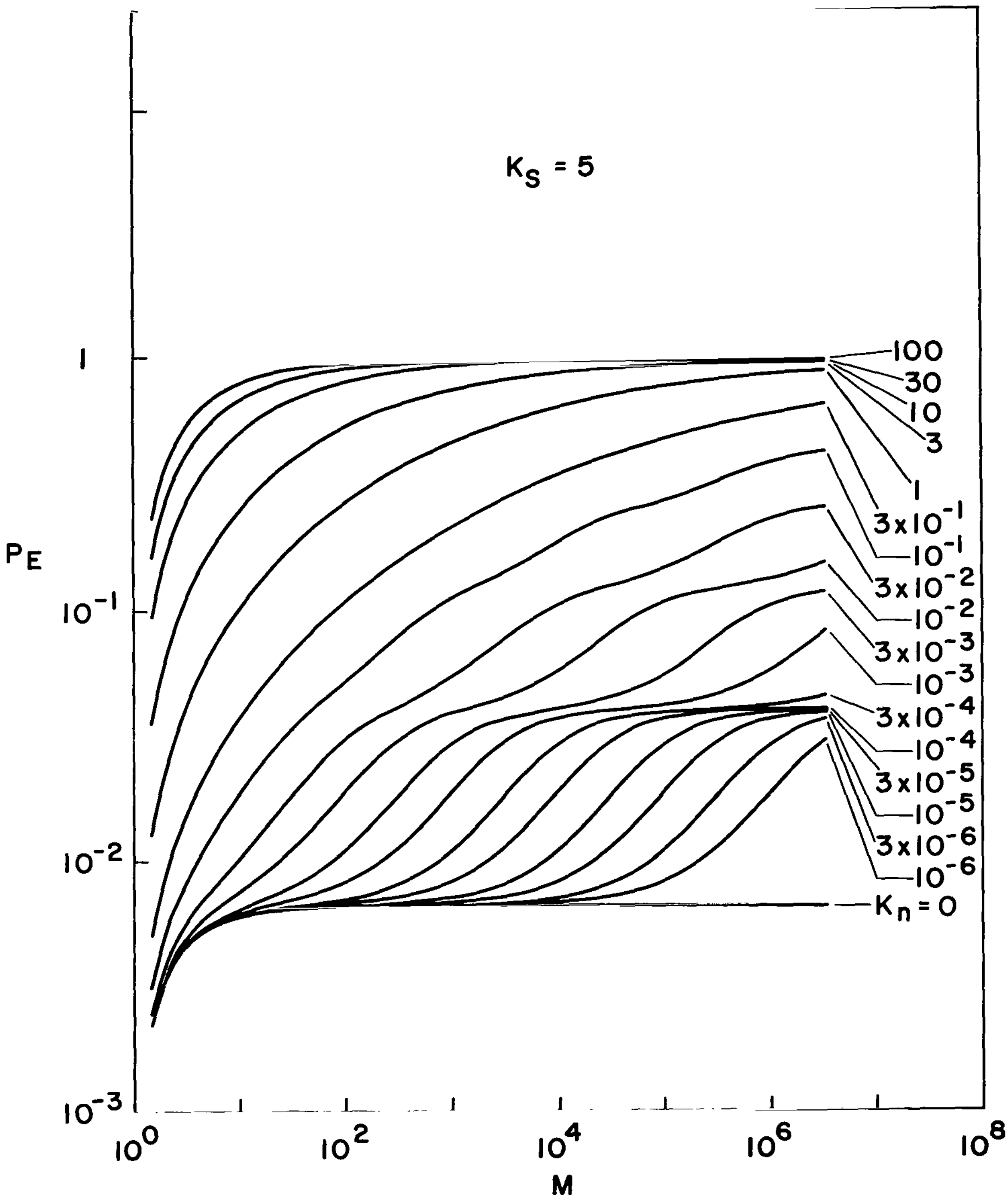

Figure 1.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

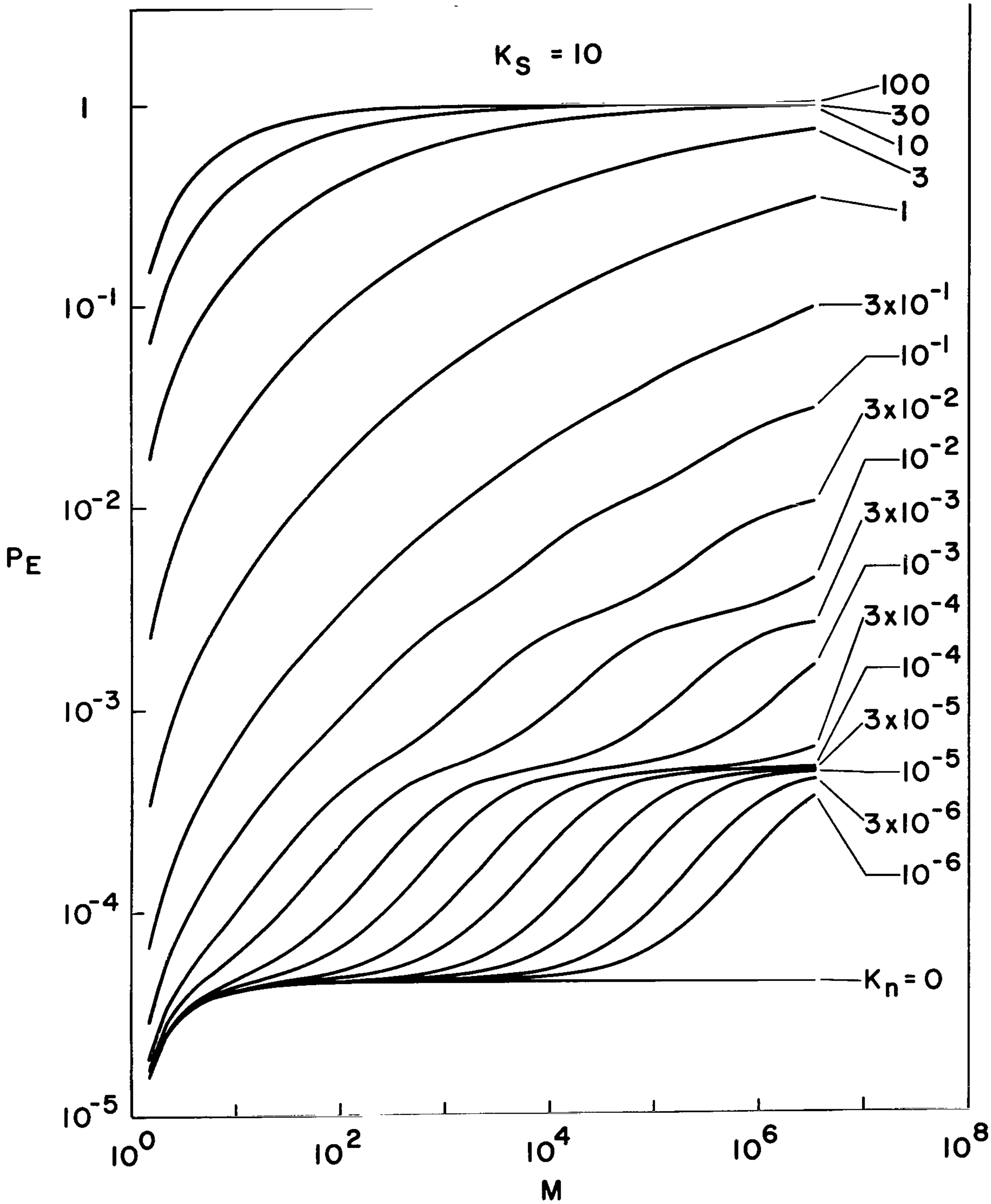

Figure 2.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

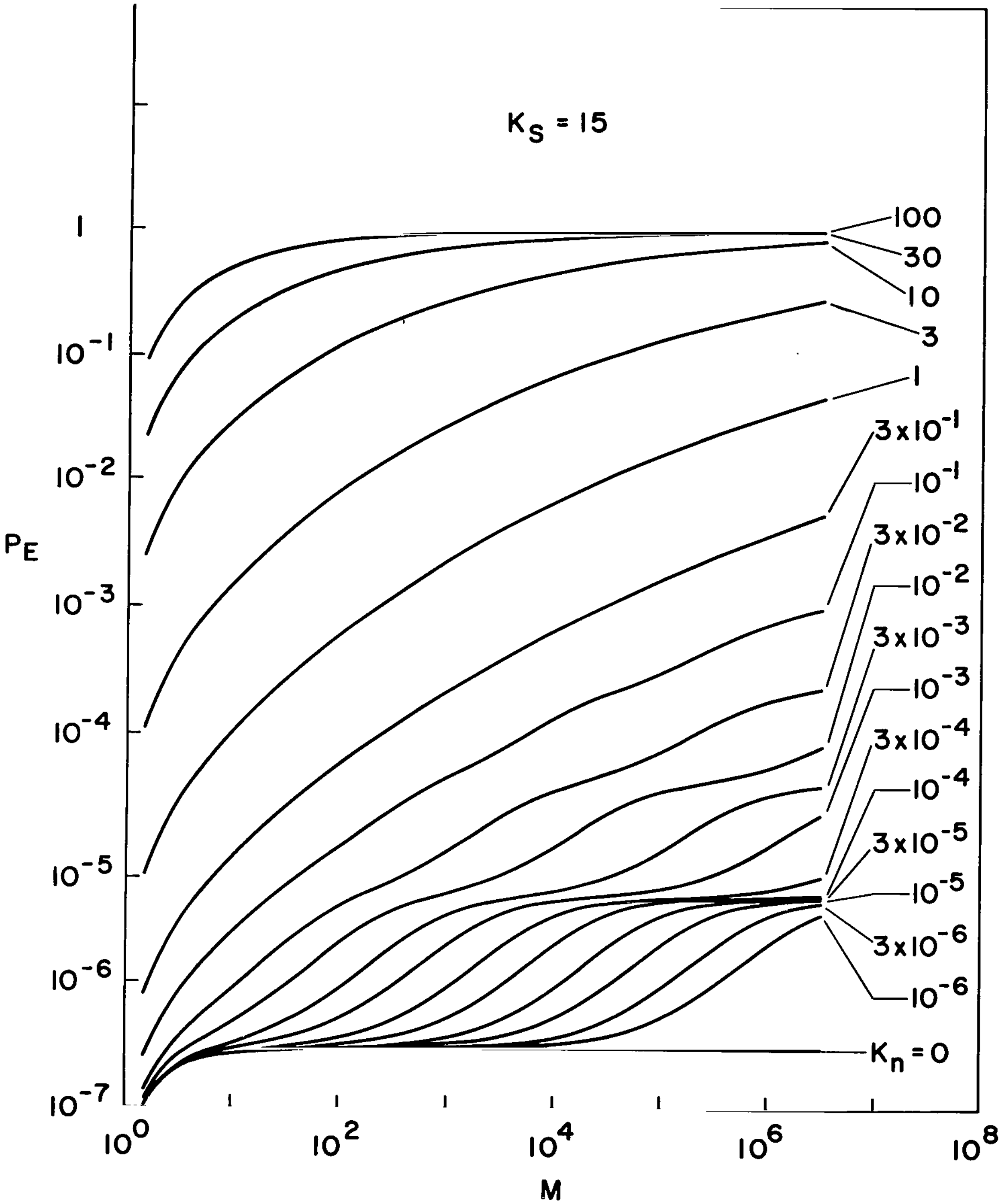

Figure 3.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

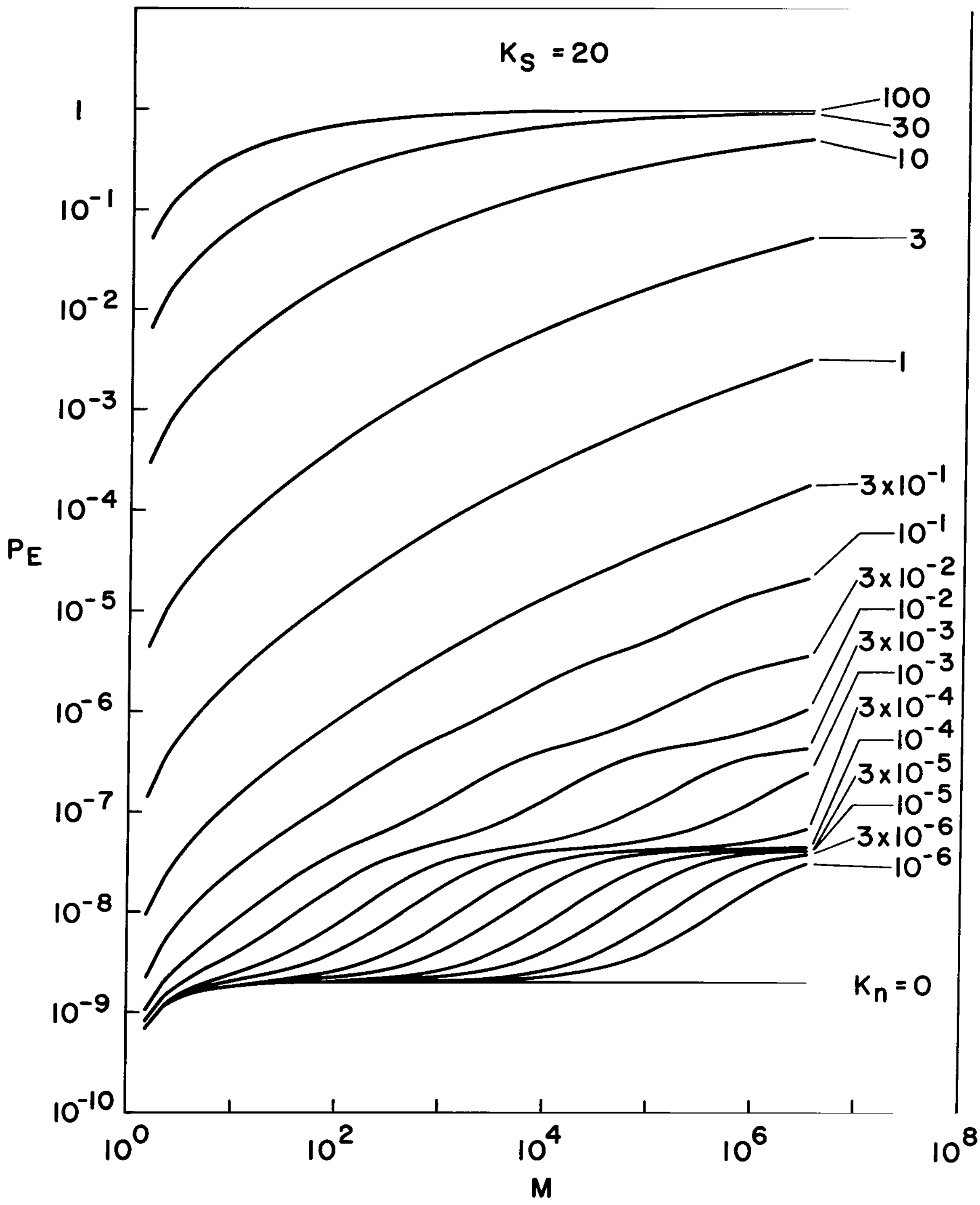

Figure 4.- Error probabilities for $\bar{n}_n \Delta T = K_n$ fixed as a function of M

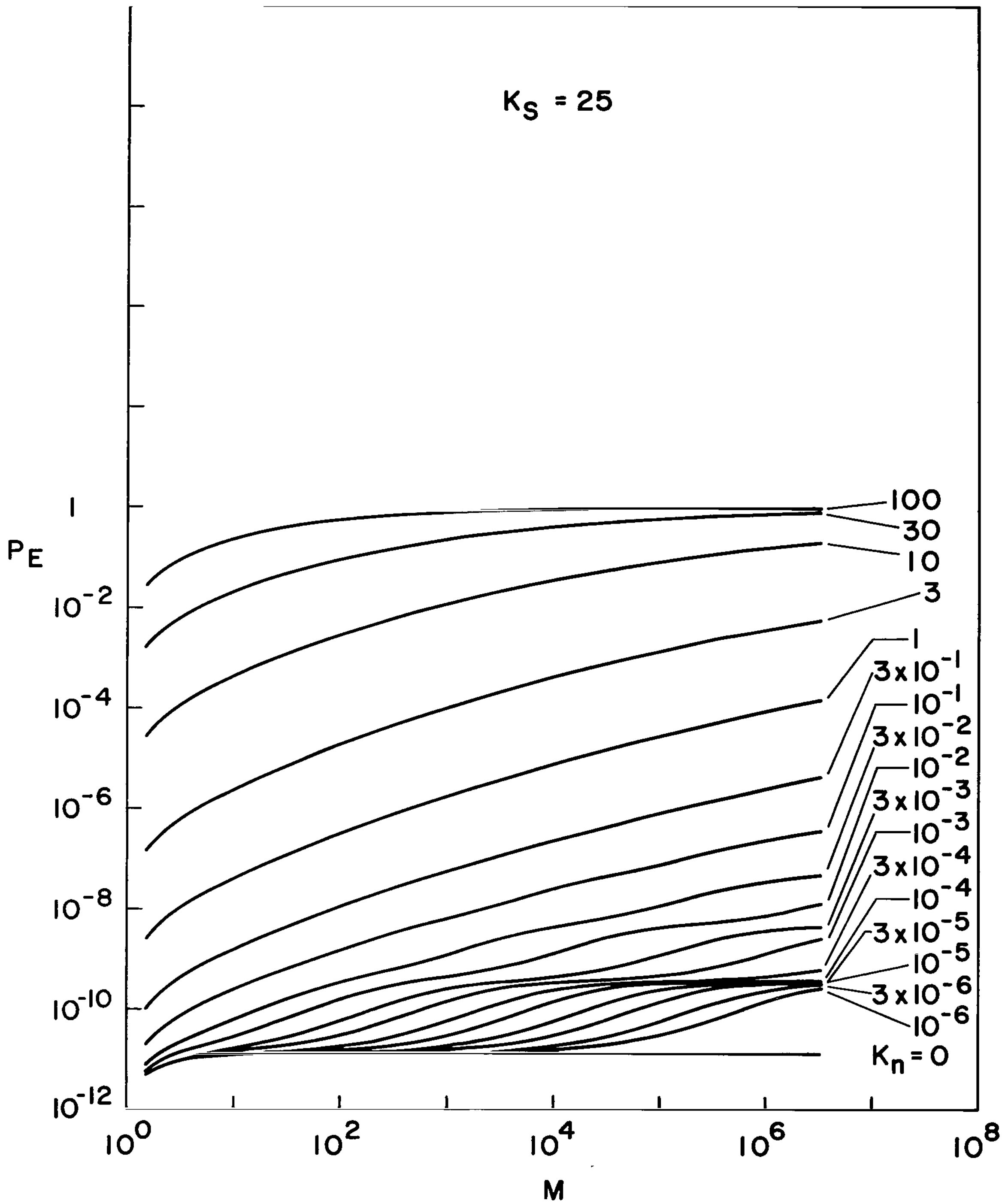

Figure 5.- Error probabilities for $\bar{n}_n \Delta T = K_n$ fixed as a function of M

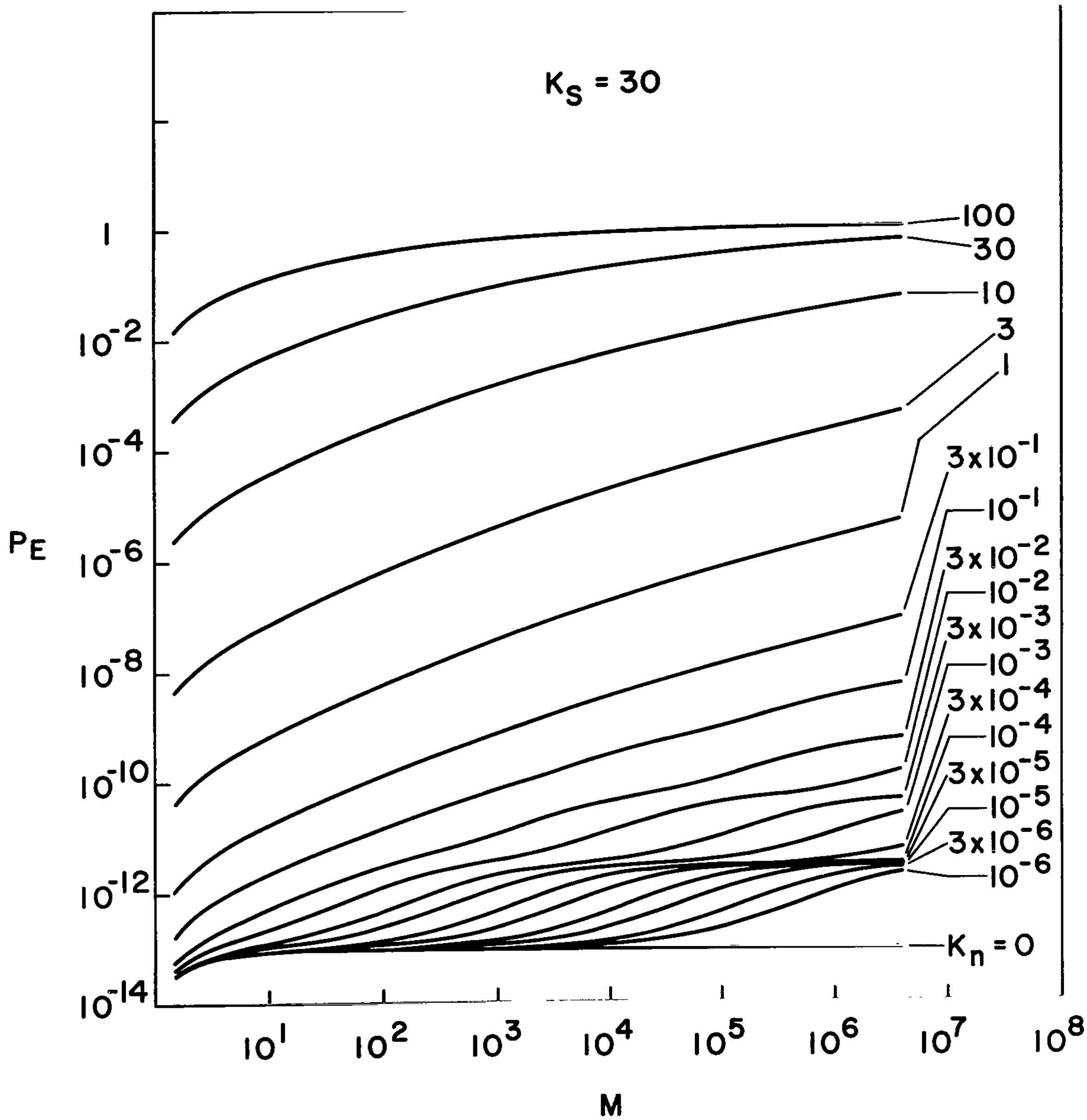

Figure 6.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

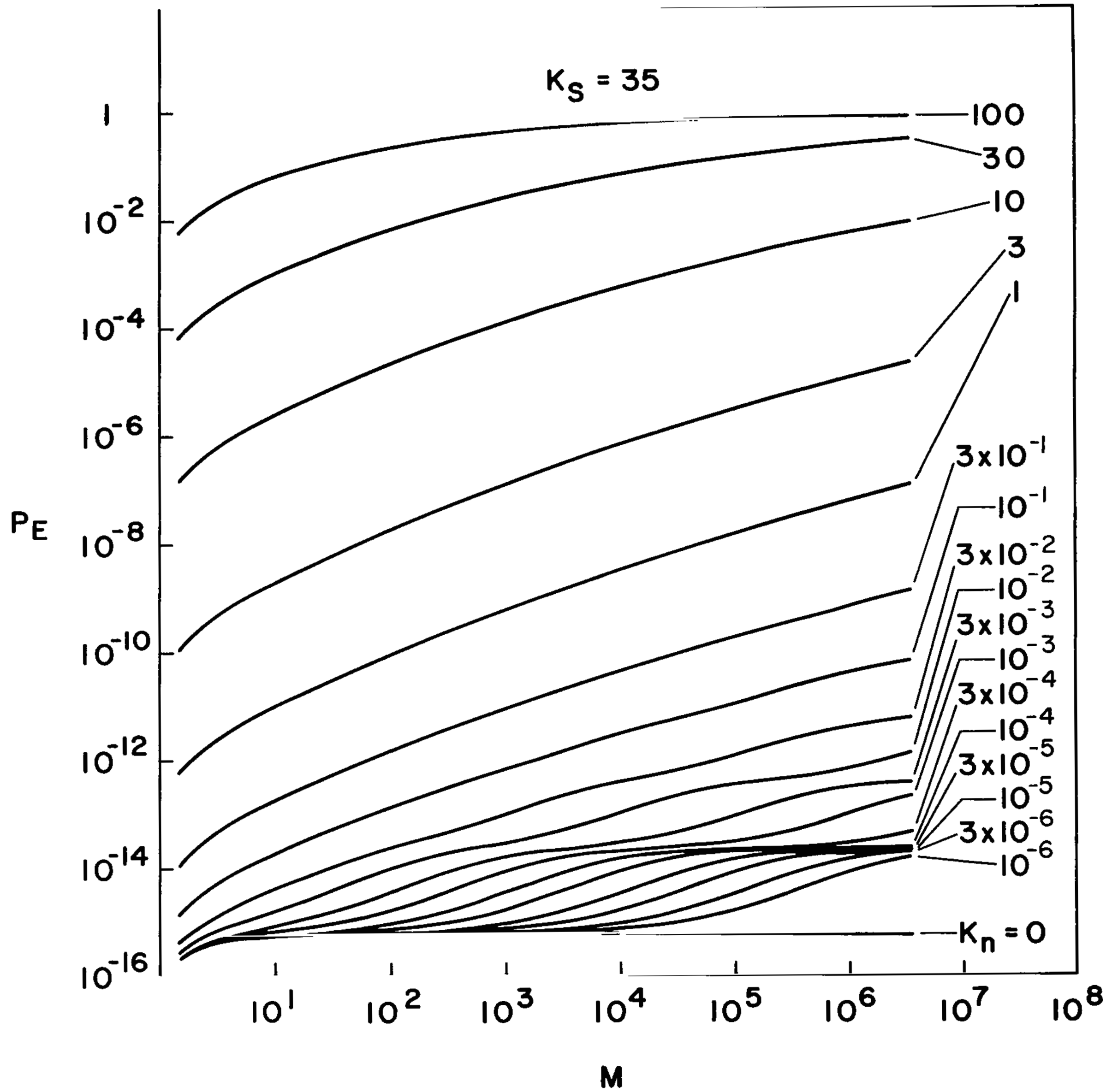

Figure 7.- Error probabilities for $\bar{n}_n \Delta T = K_n$ fixed as a function of M

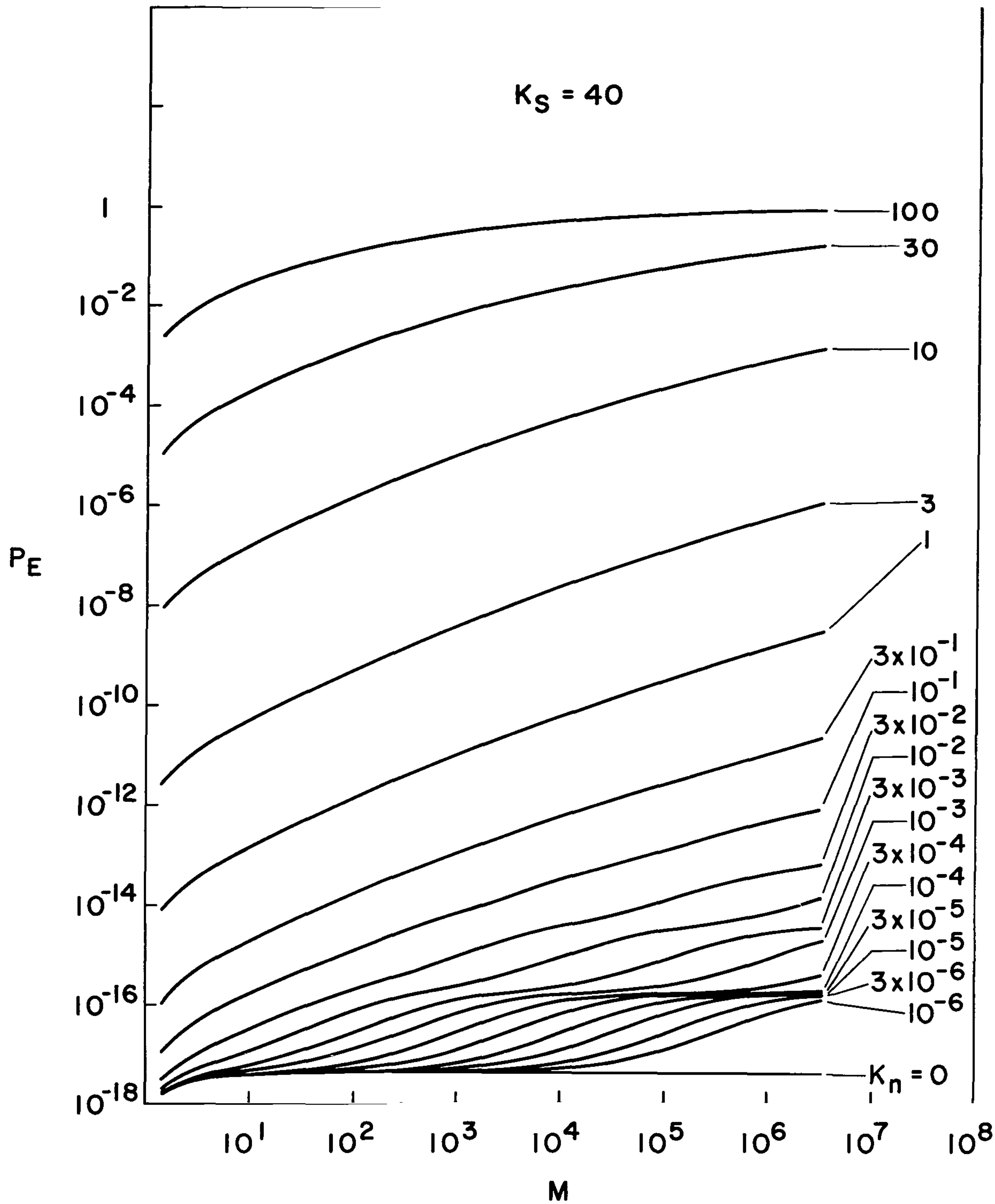

Figure 8.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

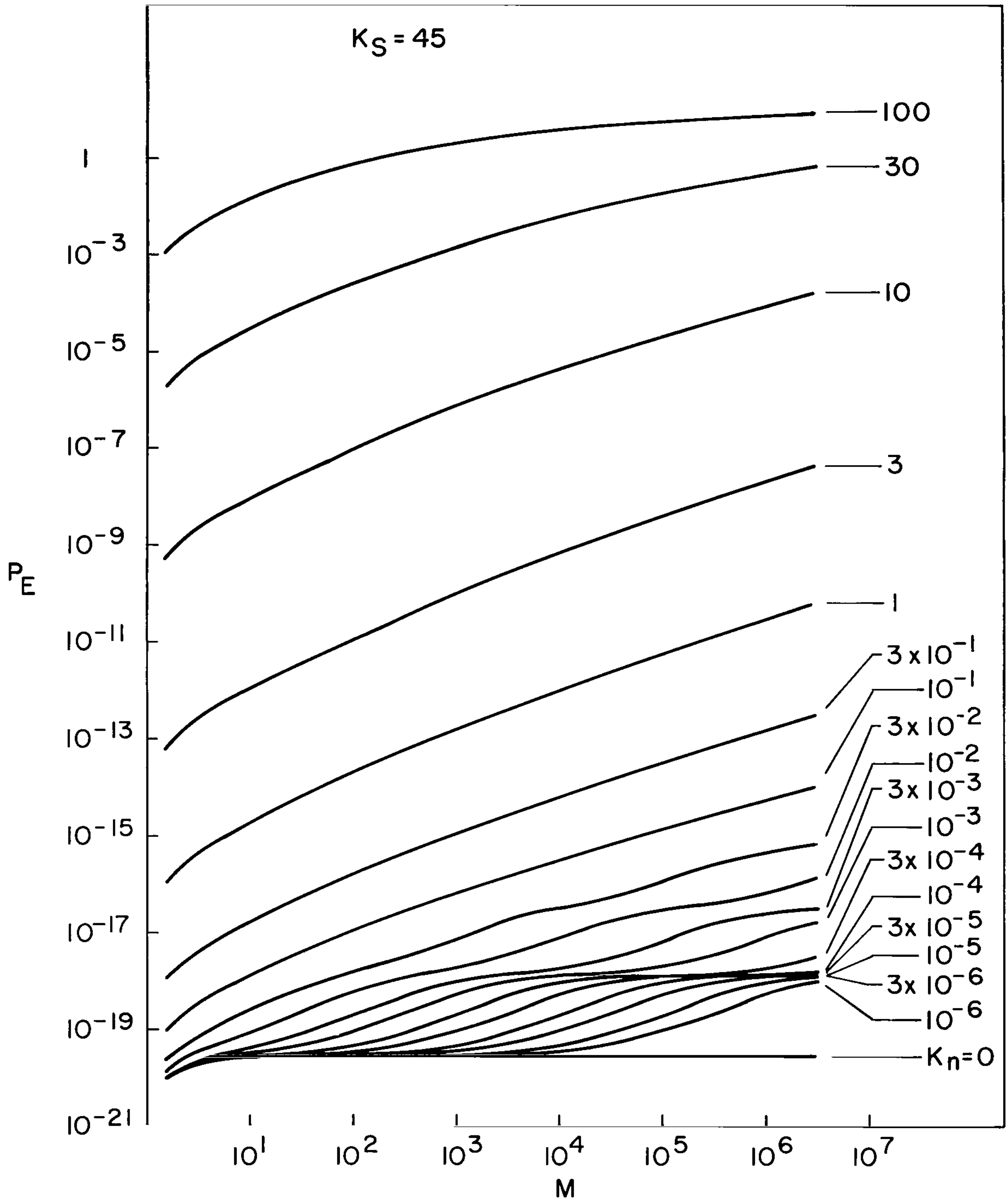

Figure 9.- Error probabilities for $\overline{n}_n \Delta T = K_n$ fixed as a function of M

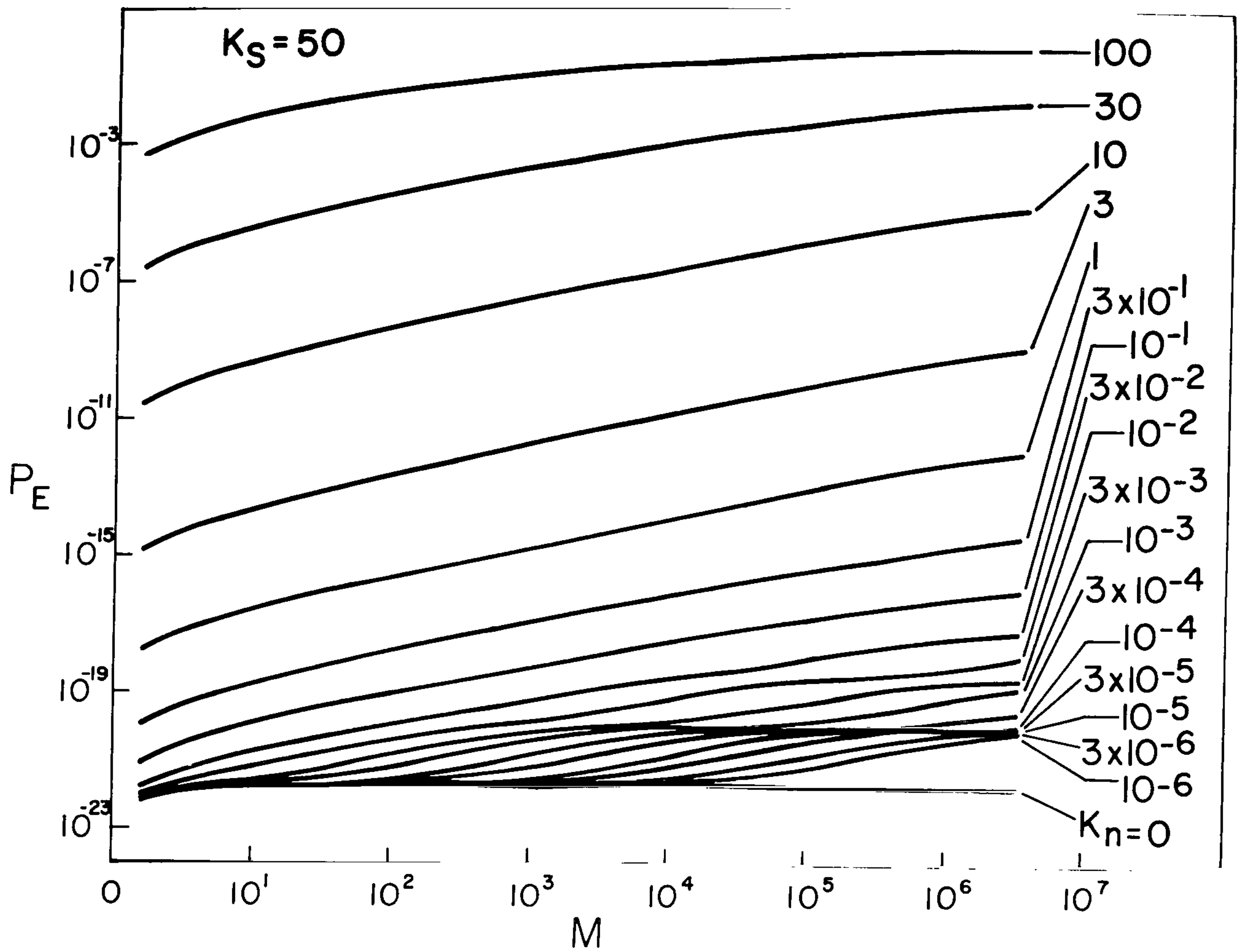

Figure 10.- Error probabilities for $\bar{n}_n \Delta T = K_n$ fixed as a function of M

where $\Delta\lambda$ is in angstroms, α is the resolution in arc seconds, and D is the diameter of the collector in centimeters. For diffraction-limited optics, α is related to D by

$$\alpha = 2.44 \times 10^5 \frac{\lambda}{D} \text{ arc seconds;}$$

hence

$$K_n = 4.76\,\eta\,\Delta T\,T_o(\Delta\lambda)\lambda^2 \times 10^{11}$$

with λ the wavelength in centimeters.

In the second form (Program 2, Figures 11-20) we consider the sampling interval T to be a constant and examine P_E as M becomes large, or since $\Delta T = T/M$, as ΔT becomes small. The assumption is that the signal energy can always be concentrated in the ΔT interval. For this case we use the parameters K_s and $K = \bar{n}_n T$. The noise in the ΔT interval is then K/M . K_s can be calculated as before while K is calculated as

$$K = \eta \frac{P_n T}{hf} .$$

It can be shown analytically that as $M \rightarrow \infty$, an asymptotic value for P_E is reached, where (ref. 5)

$$\lim_{M \rightarrow \infty} P_E = \left[1 + K_s \left[\frac{K - 1 + e^{-K}}{K}\right]\right] e^{-K_s} .$$

This asymptote is quite apparent in Figures 11-20 and is seen to vary with K as expected. Physically, this implies that if bandwidth and computation are expedient, one can always approach

$$P_E = (1 + K_s)\, e^{-K_s}$$

independent of the noise background by using narrower intervals. This procedure also provides better range resolution for the radar case. Implicit in these calculations is the fact that $\bar{n}_n$ is a constant or that the optical filter bandwidth is generally quite large.

PROGRAM 2

```
      04/30/68                      80/80 LIST
    HURW2,C76501,T500,CM50000.  NASA  I01074  KARP4   HURWITZ
     REQUEST TAPE7,LO.     PLOT SAO 766 RING IN.
    RFL,50000.
    RUN(S,,,,,,77777)
    LGO.

          PROGRAM KARP4 (INPUT,OUTPUT,TAPE7,TAPE5=INPUT,TAPE6=OUTPUT)
          DIMENSION EX(100),EXK(100),NPT(100),WY(100),X(3000),Y(3000)
          DOUBLE PRECISION A,B,EKSN,EL10,ELM,EM,EXK,PD,PE,PI,PX,
         1     S,SAVE,SUM,SUMPI,T,TERM,TERM2,TERM3,TEST,TOTAL,X1,
         2     XK,XKN,XKNMAX,XKNMIN,XKS,XKSN,XM,XM1,XMB,XMBMIN,XX
          DATA IN,NOUT,IPLT,TEST/5,6,7,1.D-24/
          DATA XAX,YAX/11.,14./
          DATA IXMAX,XKNMAX,XKNMIN,XMBMIN/1000,3.D+2,1.D-30,1.D-12/
    CCC   KARP/HURWITZ
    CCC   A=ABS.VALUE OF DIFFERENCE BETWEEN TWO CONSECUTIVE TERMS
    CCC   B DESCRIBED IN WRITE UP
    CCC   EKSN=EXP(-XKSN)
    CCC   EL10=CONSTANT USED FOR TESTING MAGNITUDES OF NUMBERICAL VALUES
    CCC   ELM=LOG(M)
    CCC   EM=INITIAL VALUE OF M
    CCC   EXK=INPUT OF K
    CCC   EX=ARRAY FOR LOG(M) FOR EACH KS AND EACH K
    CCC   I=INDEX
    CCC   I1 COUNTS NUMBER TOTAL NUMBERS OF PE'S
    CCC   ICUTOF USED TO INSURE NO PREMATURE CUT-OFF OF THE CALCULATION
    CCC   IEND=TEST VALUE FOR STOPPING RUN
    CCC   IEND MUST BE NINE FOR ALL BUT LAST DATA CARD
    CCC   IXMAX=MAX VALUE FOR INDEX IX
    CCC   J=INDEX
    CCC   JMP=TEST VALUE FOR READING VALUES OF K
    CCC   JMP MUST BE ZERO FOR FIRST DATA CARD
    CCC   NK=NUMBER OF VALUES OF K
    CCC   NM=NUMBER OF VALUES OF M
    CCC   NN=NPT(I)
    CCC   NPT(J) COUNTS NUMBER OF M'S FOR EACH K
    CCC   N1=NN+1
    CCC   N2=NN+2
    CCC   PD DESCRIBED IN WRITE UP
    CCC   PE=1-PD
    CCC   PI=(KN**I)*EKSN/(I FACTORIAL)
    CCC   PX=XKSN*EKSN
    CCC   S USED TO TEST MAGNITUDES
    CCC   SAVE LAST VALUE OF TERM
    CCC   SUM=SUMMATION (KN**I/I)
    CCC   SUMPI=SUMMATION (PI)
    CCC   T=DUMMY
    CCC   TERM=PX*TERM2*TERM3
    CCC   TERM2=SUMPI**(M-1)
    CCC   TERM3=(1+B)**M-1/(M*B)
    CCC   TOTAL=SUMMATION(TERM)
    CCC   WY=ARRAY FOR LOG(PE) FOR EACH KS AND EACH K
    CCC   X=ARRAY FOR LOG(M) FOR ALL CURVES FOR EACH KS
    CCC   XK=EXK(J)
    CCC   XKN=KN
    CCC   XKNMAX=MAX VALUE OF KN
    CCC   XKNMIN=MIN VALUE OF KN
```

```
CCC   XKS=KS
CCC   XKSN=KS+KN
CCC   XM=EM**I
CCC   XMB=XM*B
CCC   XMBMIN=TEST VALUE FOR XMB
CCC   XM1=XM-1
CCC   XX=IX=INDEX OF OUTER LOOP
CCC   Y=ARRAY FOR LOG(PE) FOR ALL CURVES FOR EACH KS
      EL10=300.D0*DLOG(10.D0)
      CALL INITPLT(IPLT)
    1 READ (IN,3) EM,XKS, NM,NK,IEND,JMP
      L=0
      IF (JMP .NE.0) GO TO 2
      READ (IN,4) (EXK(I), I=1,NK)
    4 FORMAT (D10.3)
CCC   LOOP FOR K
    2 DO 70 J=1,NK
      NPT(J)=0
    3 FORMAT (2D10.3,4I5)
      XK=EXK(J)
CCC  LOOP FOR M
   10 DO 65  I=1,NM
CCC   PRINT HEADINGS FIRST TIME THROUGH THIS LOOP
      IF (I.NE.1) GO TO 14
      WRITE (NOUT,12) XKS,XK
   12 FORMAT(1H1,40X,3HKS=,D9.2,10X,2HK=,D9.2//)
      WRITE (NOUT,13)
   13 FORMAT(17X,1HM,20X,2HPD,20X,2HPE,20X,6HLOG(M),16X,7HLOG(PE)//)
   14 XM=EM**I
      XM1=XM-1.D0
      XKN=XK/XM
CCC   TEST TO INSURE EXP(KN) DOES NOT EXCEED MACHINE CAPACITY
CCC   IF KN GT XKNMAX, INCREASE M AND CONTINUE
      IF (XKN.GT.XKNMAX) GO TO 65
      XKSN=XKS+XKN
CCC   TEST VALUE TO INSURE NO PREMATURE CUTOFF
      ICUTOF=2.D0*XKSN
      EKSN=DEXP(-XKSN)
CCC   FORMATION OF PX FOR X=1
      PX=XKSN*EKSN
      SUM=1.D0
      T=1.D0
CCC   FORMATION OF B FOR X=1
      B=XKN
      XMB=XM*B
CCC   TEST FOR (1+B)**M TOO LARGE
      S=DLOG(1.D0+B)
      S=EL10/XM-S
      IF (S.GT.0.D0) GO TO 17
CCC   VALUE FOR (1+B)**M TOO LARGE FOR MACHINE
      TERM3=0.D0
      GO TO 16
CCC   VALUE FOR (1+B)**M SUFFIENTLY SMALL
   17 TERM3=((1.D0+B)**XM-1.D0)/XMB
CCC   TEST FOR KN NEAR ZERO
   15 IF (XKN.GE.XKNMIN) GO TO 16
      TERM2=1.D0
```

```
      TERM3=1.D0
      GO TO 20
CCC   FORMATION OF PI,SUMPI,TERM2 FOR X=1
   16 PI=DEXP(-XKN)
      SUMPI=PI
      TERM2=PI**XM1
CCC   LOOP FOR SUMMATION OF TERM
   20 TERM=PX*TERM2*TERM3
      TOTAL=TERM
      DO 50 IX=2,IXMAX
      SAVE=TERM
      XX=IX
      X1=XX-1.D0
CCC   FORMATION OF PX FOR X.GE.2
      PX=PX*XKSN/XX
CCC   TEST FOR KN NEAR ZERO
      IF (XKN.LT.XKNMIN) GO TO 45
CCC   FORMATION OF PI FOR X.GE.2
      PI=PI*XKN/X1
CCC   FORMATION OF TERM2 FOR X.GE.2
CCC   FORMATION OF B FOR X.GE.2
      SUMPI=SUMPI+PI
   25 TERM2=(SUMPI)**XM1
      T=T*XKN/X1
      A=SUM
      SUM=SUM+T
      B=XKN*B*A/(XX*SUM)
CCC   TEST FOR (1+B)**M TOO LARGE
      S=DLOG(1.D0+B)
      S=EL10/XM-S
      IF (S.LT.0.D0) GO TO 50
CCC   FORMATION OF TERM3
CCC   TEST FOR XMB LE XMBMIN
CCC   IF MB LE XMBMIN APPROX TERM3 WITH FIRST TWO TERMS OF MB ONLY
   35 XMB=XM*B
      IF (XMB.GT.XMBMIN) GO TO 40
      TERM3=     1.D0+(XM-1.D0)*B/2.D0
      GO TO 45
   40 TERM3=((1.D0+B)**XM-1.D0)/XMB
CCC   FORMATION OF TERM FOR X.GE.2
   45 TERM=PX*TERM2*TERM3
CCC   SUMMATION OF TERMS
      TOTAL=TOTAL+TERM
CCC   TEST IF DIFFERENCE OF TERMS FOR X=N AND X=N+1 IS SUFF. SMALL
      A=DABS(TERM-SAVE)
      IF (A.LT.TEST.AND.IX.GT.ICUTOF) GO TO 55
CCC   TEST TO INSURE AGAINST PREMATURE CUTOFF
   50 CONTINUE
CCC   COMPUTATION OF FINAL VALUE FOR A GIVEN M
   55 PD=TOTAL+DEXP(-(XKS+XM*XKN))/XM
      PE=1.D0-PD
      L=L+1
      ELM=DLOG10(XM)
      X(L)=SNGL(ELM)
      ELM=DLOG10(PE)
      Y(L)=SNGL(ELM)
      NPT(J)=NPT(J)+1
```

```
      WRITE (NOUT,60) XM,PD,PE,X(L),Y(L)
   60 FORMAT(9X,3(D16.9,6X),2(F16.9,6X))
   65 CONTINUE
   70 CONTINUE
      L1=L+1
      L2=L+2
      X(L1)=0.
      X(L2)=0.
      Y(L1)=0.
      Y(L2)=0.
CCC   PLOTTING ROUTINES FOR CAL-COMP PLOTTER
      CALL PLOT(0.,-31.,-3)
      CALL PLOT(0.,2.,-3)
      CALL SCALE (X,XAX,L,1)
      CALL SCALE (Y,YAX,L,1)
      CALL AXIS (0.,0.,6HLOG(M),-6,XAX,0.,X(L1),X(L2))
      CALL AXIS (0.,0,7HLOG(PE),7,YAX,90.,Y(L1),Y(L2))
      I1=0
      DO 90  I=1,NK
      NN=NPT(I)
      DO 85  J=1,NN
      I1=I1+1
      EX(J)=X(I1)
   85 WY(J)=Y(I1)
      N1=NN+1
      N2=NN+2
      EX(N1)=X(L1)
      EX(N2)=X(L2)
      WY(N1)=Y(L1)
      WY(N2)=Y(L2)
      CALL LINE(EX,WY,NN,1,0,0)
   90 CONTINUE
      CALL SYMBOL(0.5,16.0,.3,3HKS=,0.,3)
      CALL NUMBER(1.5,16.0,.3,XKS,0.,-1)
      CALL PLOT(20.,0.,-3)
  110 CONTINUE
      IF(IEND.EQ.9) GO TO 1
      CALL FIN(IPLT)
  120 STOP
      END
```

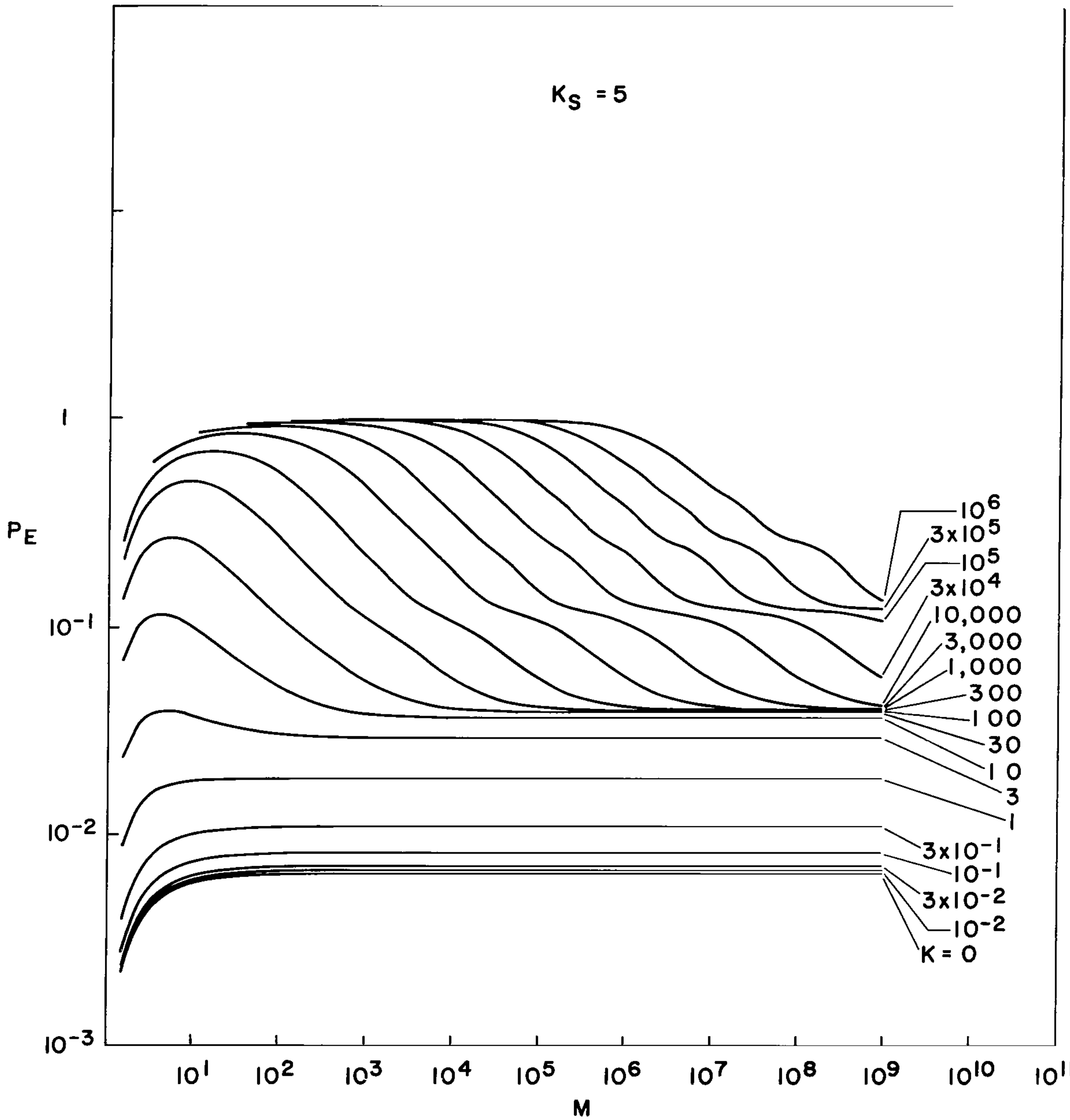

Figure 11.- Error probabilities for $\overline{n}_n T = K$ fixed as a function of M

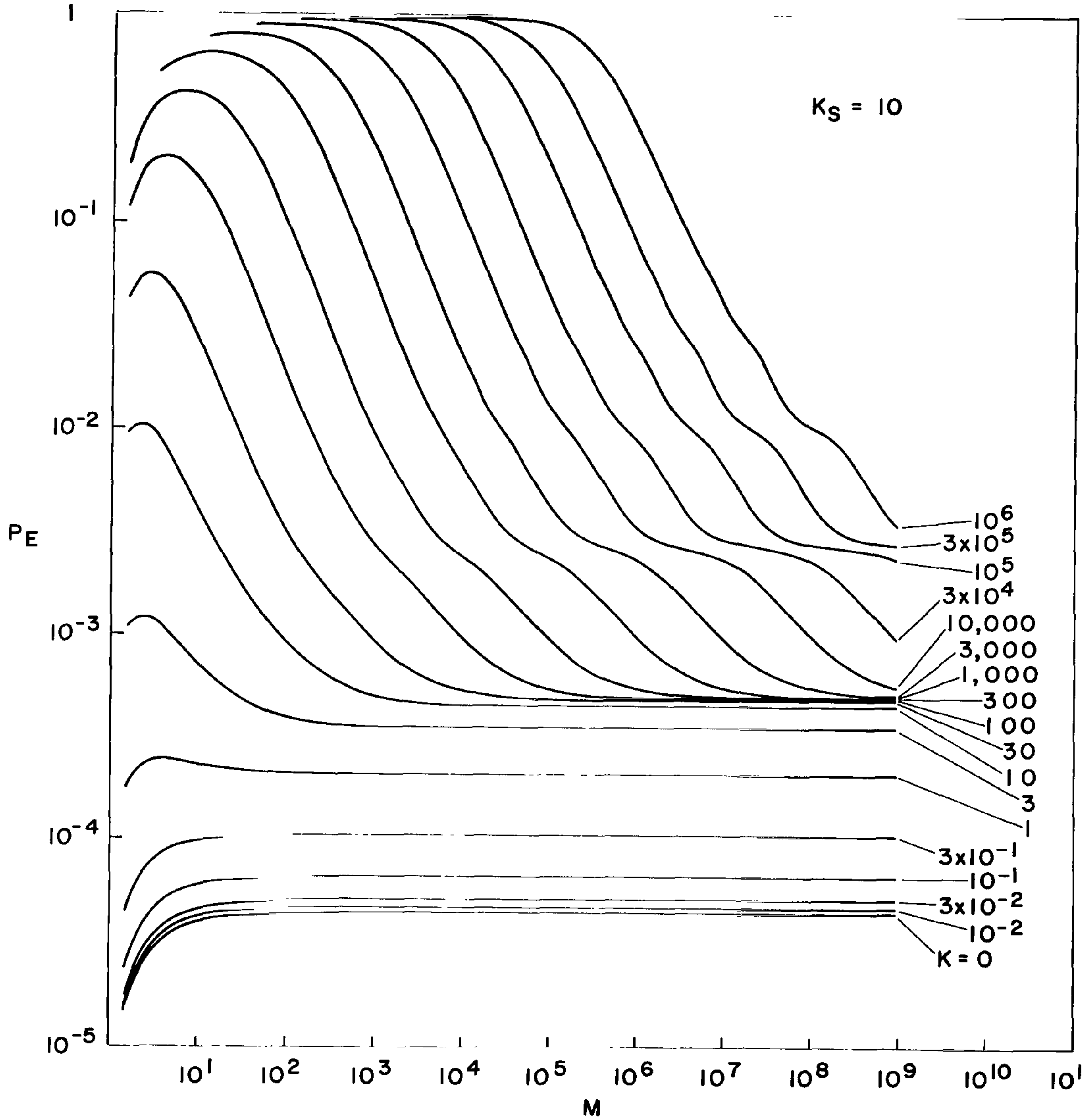

Figure 12.- Error probabilities for $\bar{n}_n T$ = K fixed as a function of M

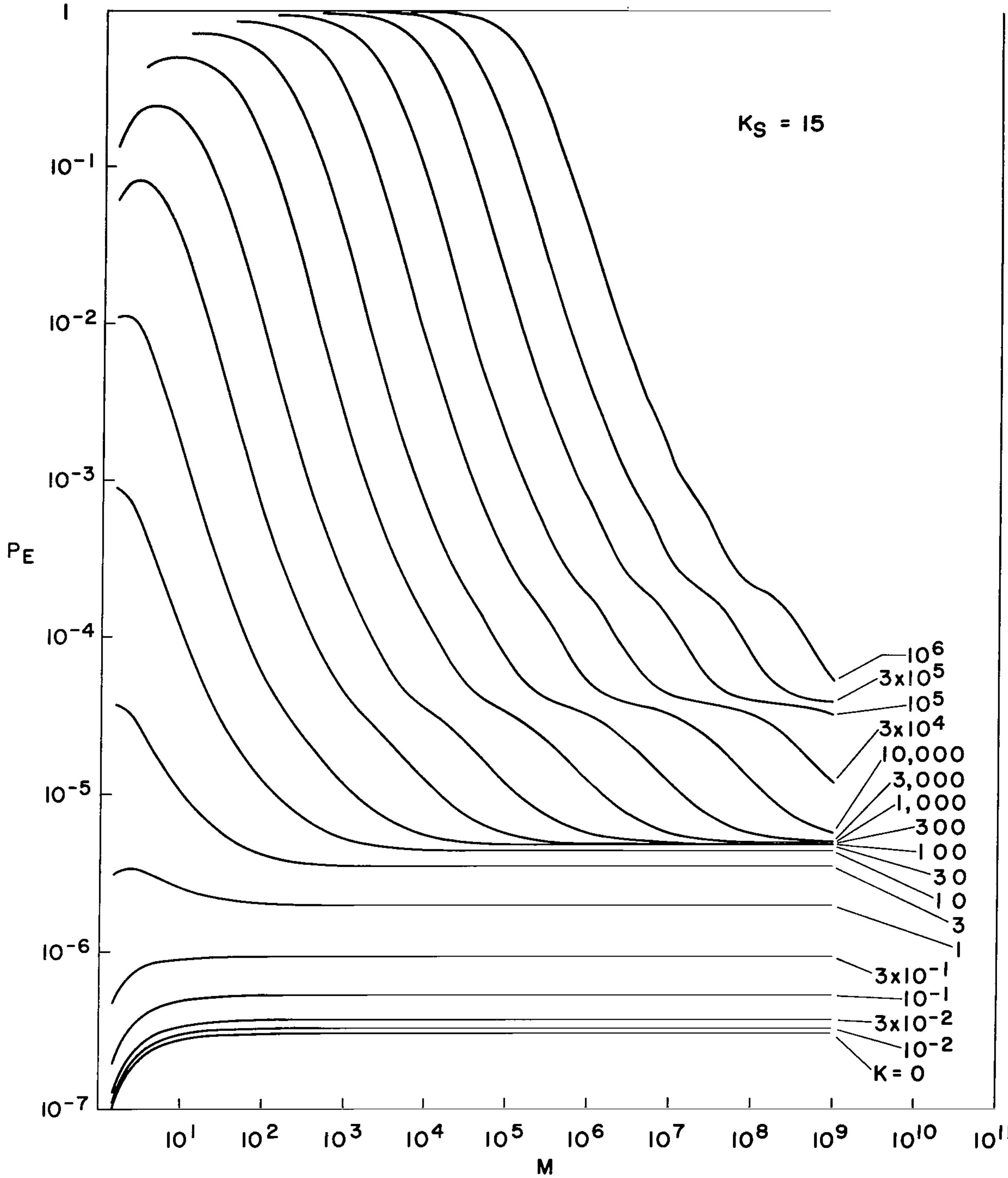

Figure 13.- Error probabilities for $\bar{n}_n T = K$ fixed as a function of M

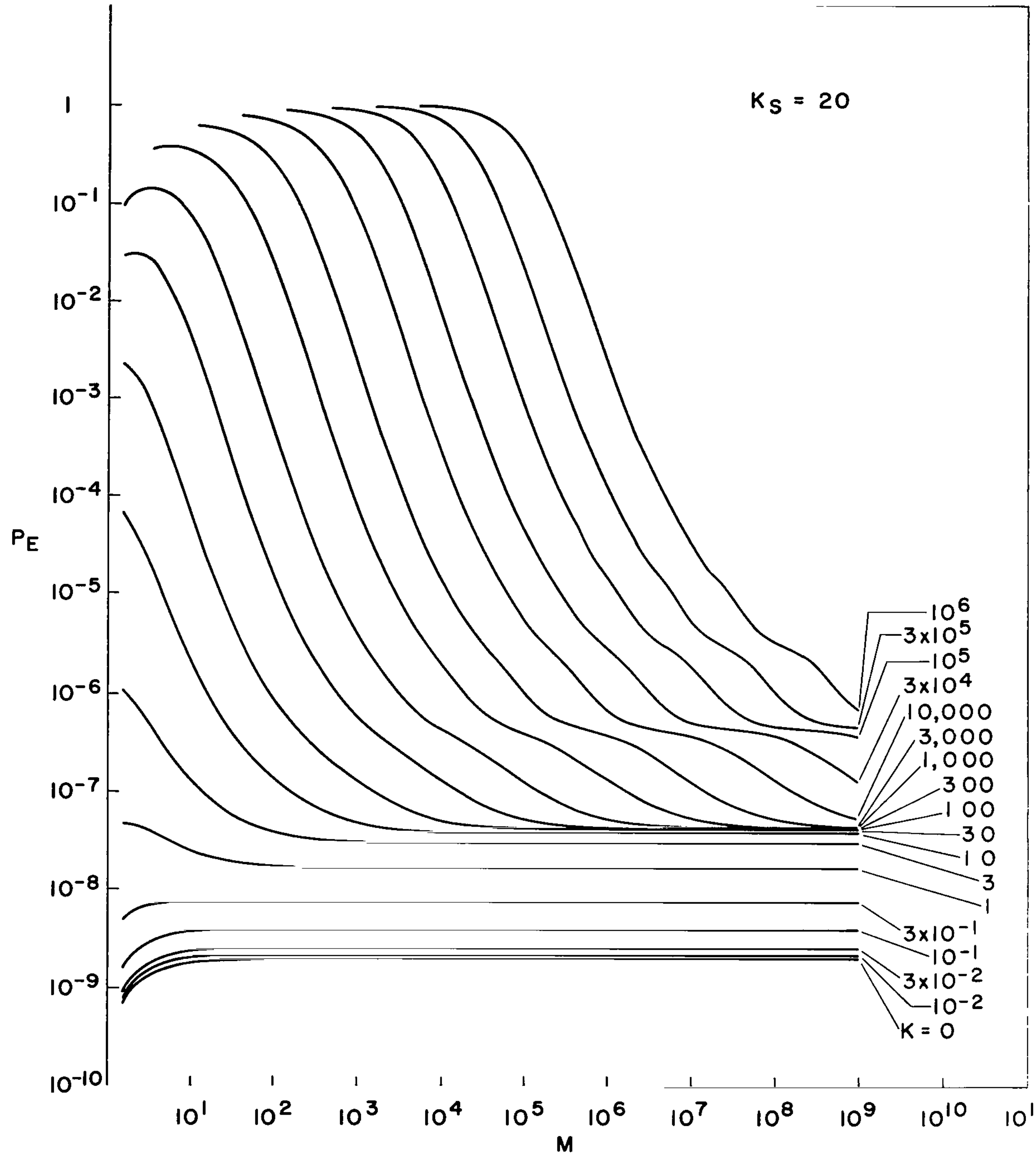

Figure 14.- Error probabilities for $\bar{n}_n T = K$ fixed as a function of M

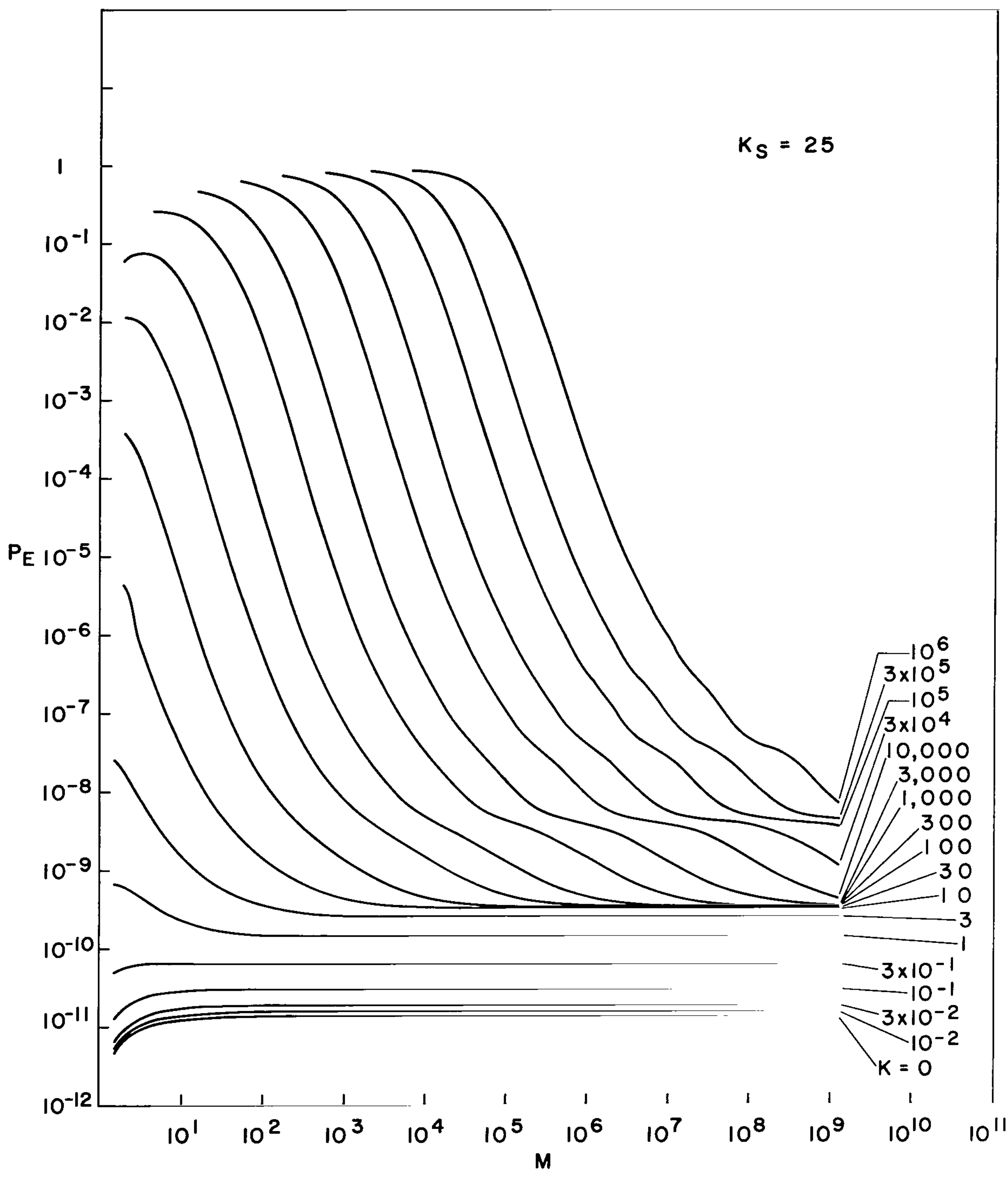

Figure 15.- Error probabilities for $\overline{n}_n T = K$ fixed as a function of M

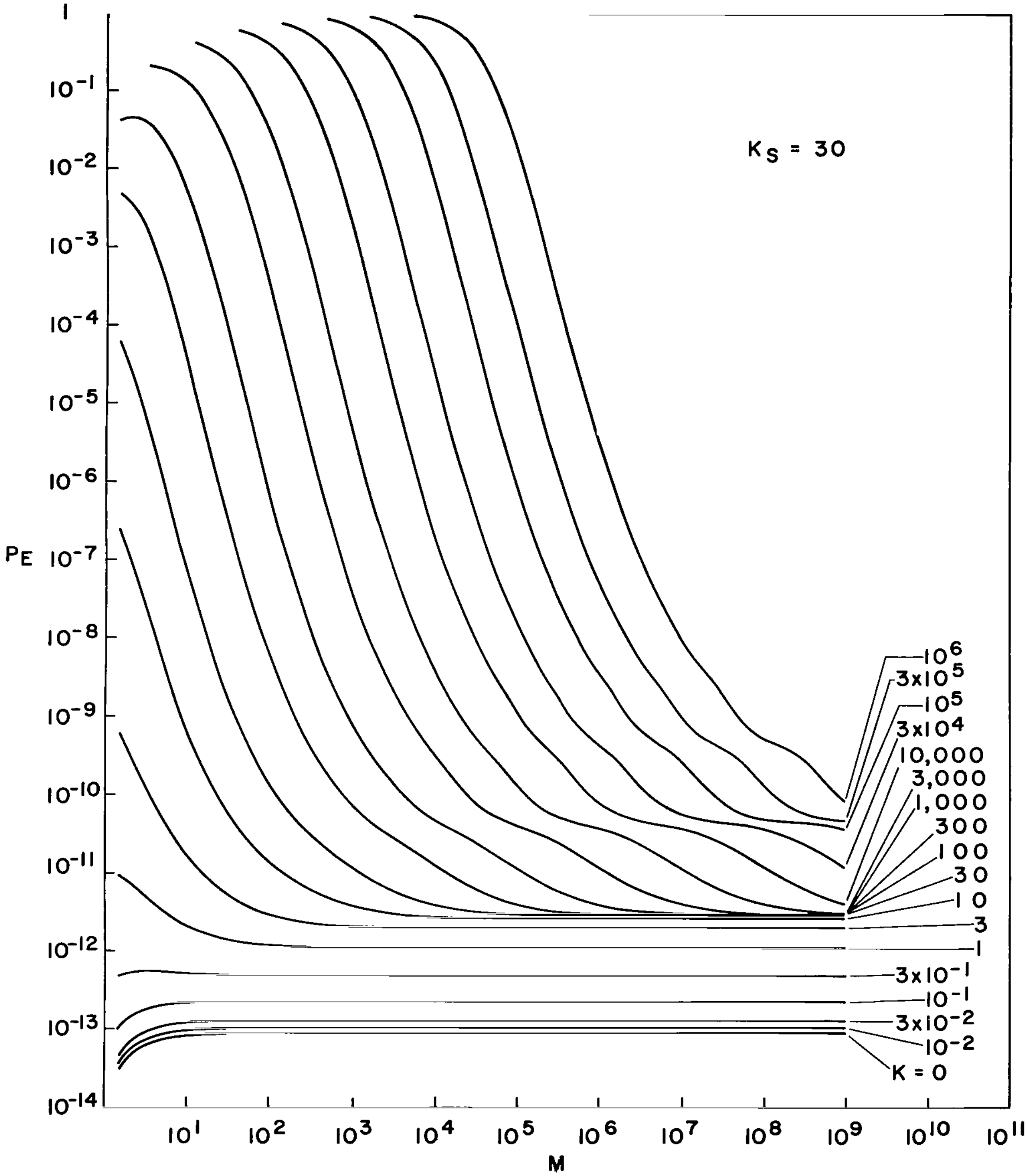

Figure 16.- Error probabilities for $\overline{n}_n T = K$ fixed as a function of M

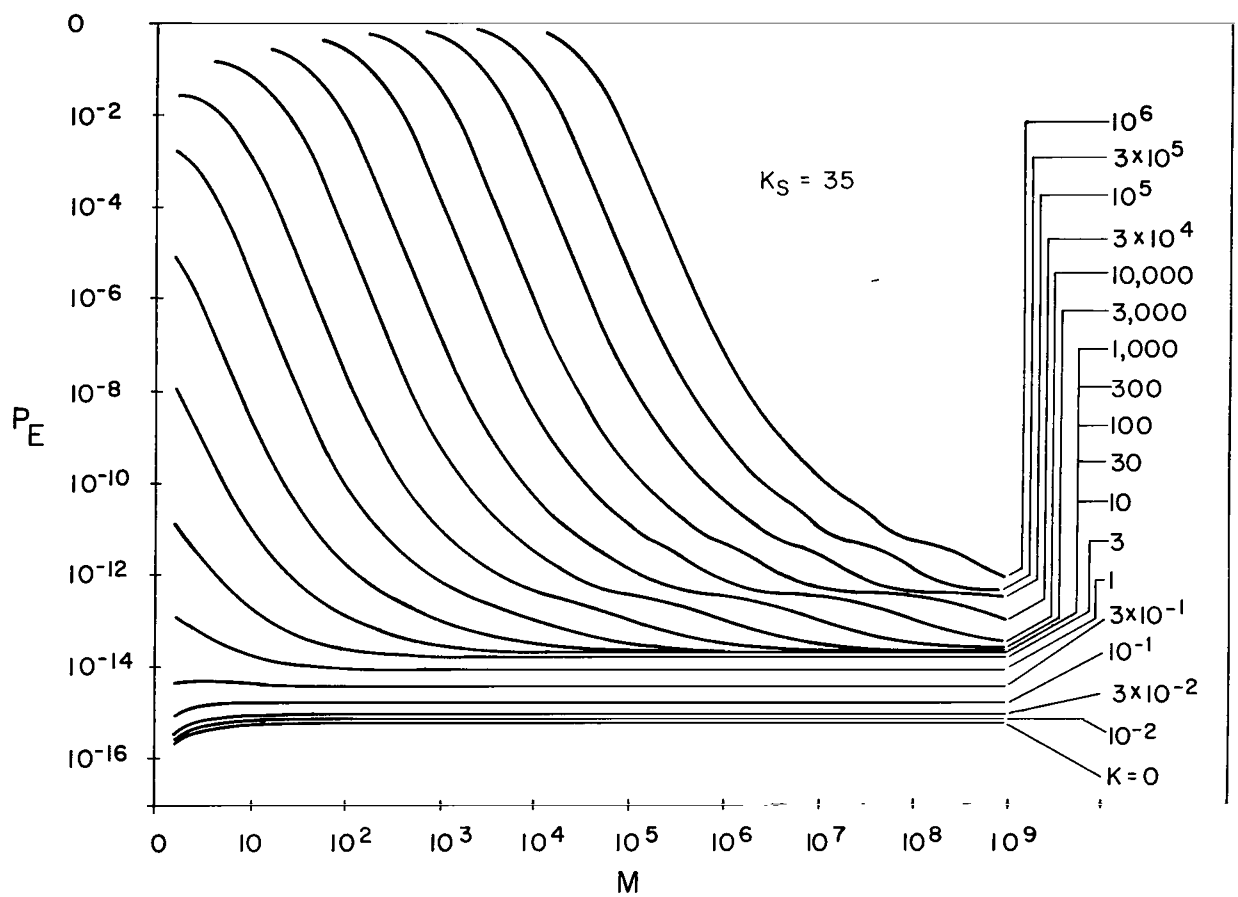

Figure 17.- Error probabilities for $\bar{n}_n T = K$ fixed as a function of M

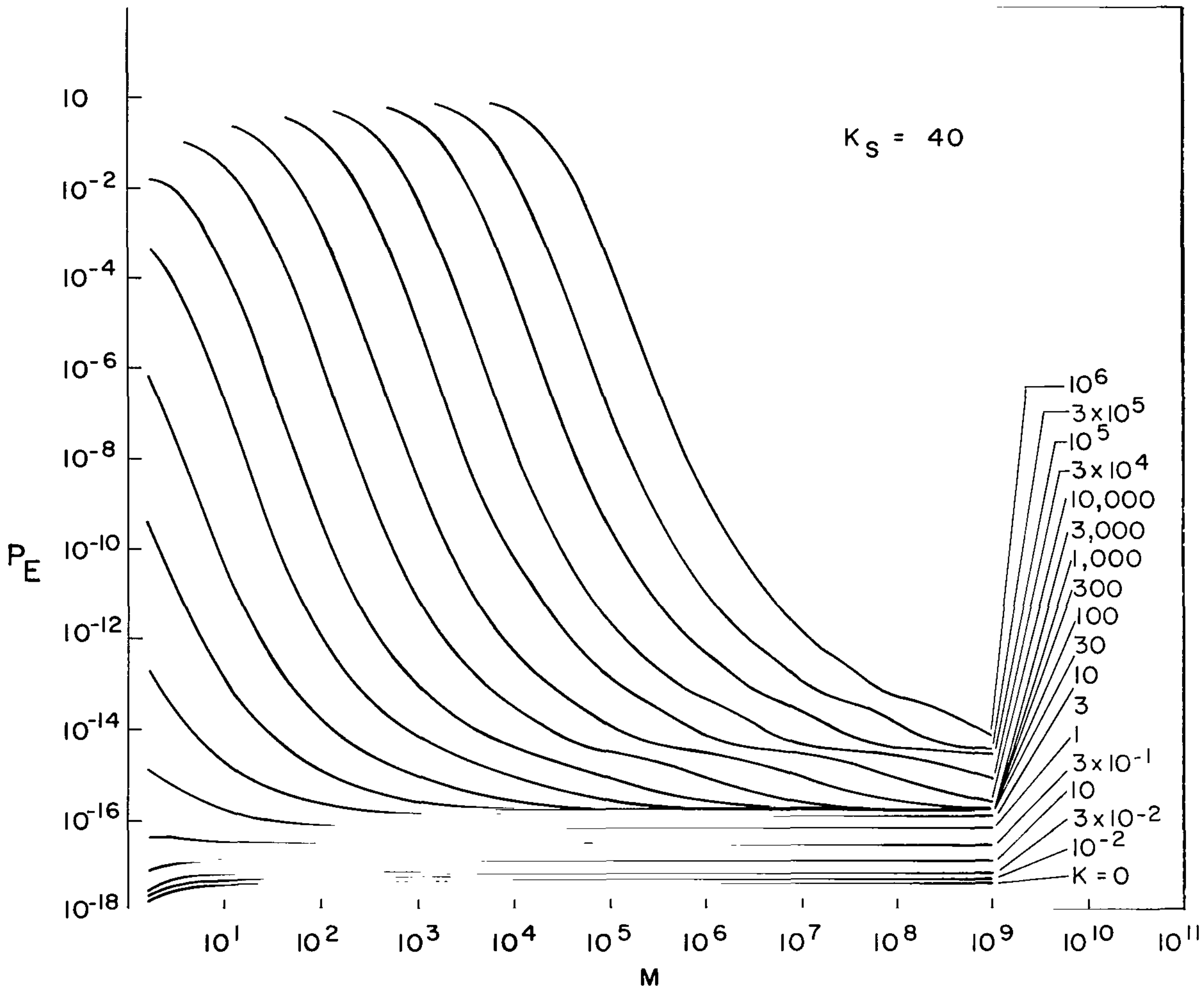

Figure 18.- Error probabilities for $\bar{n}_n T = K$ fixed as a function of M

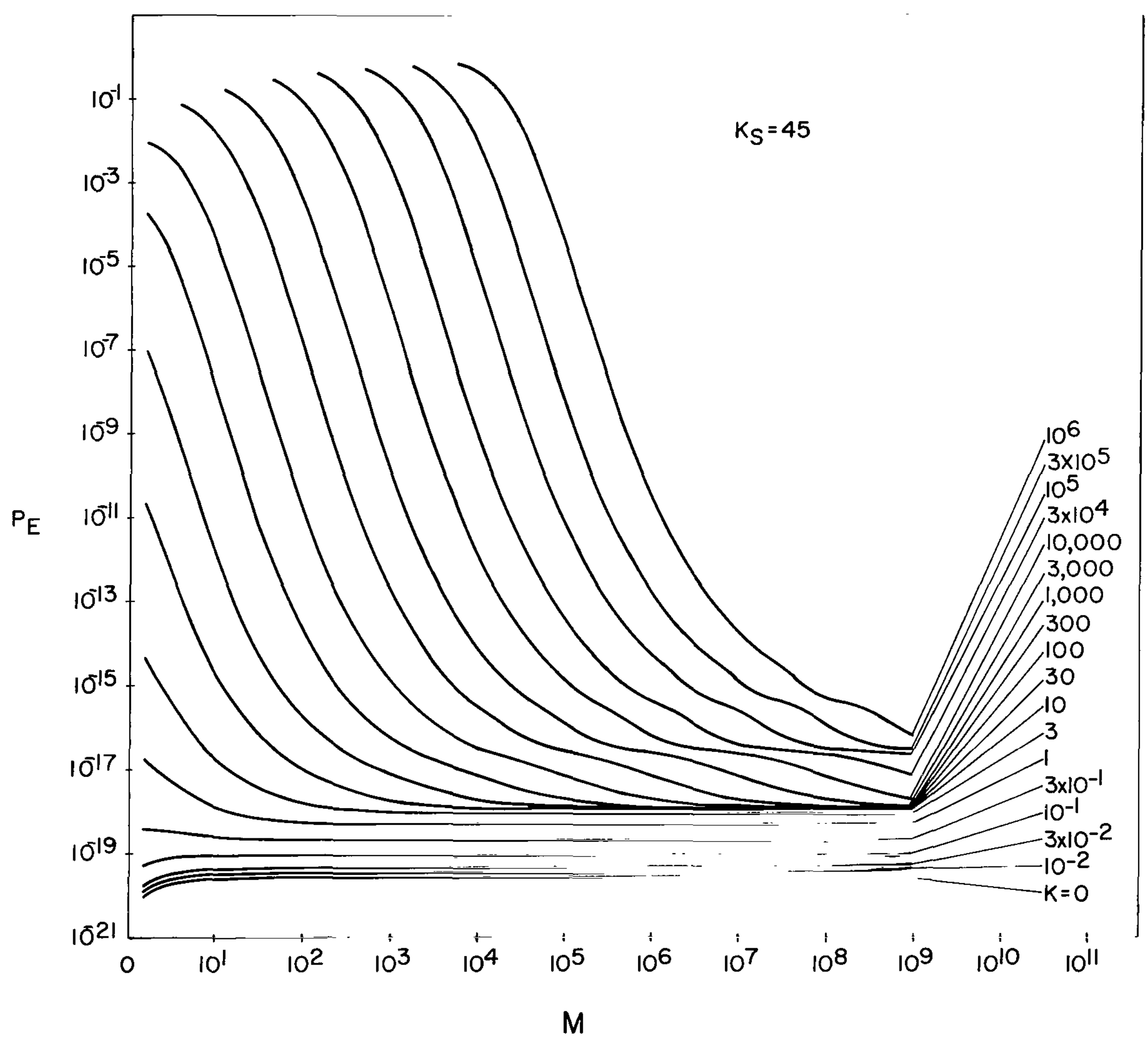

Figure 19.- Error probabilities for $\bar{n}_n T = K$ fixed as a function of M

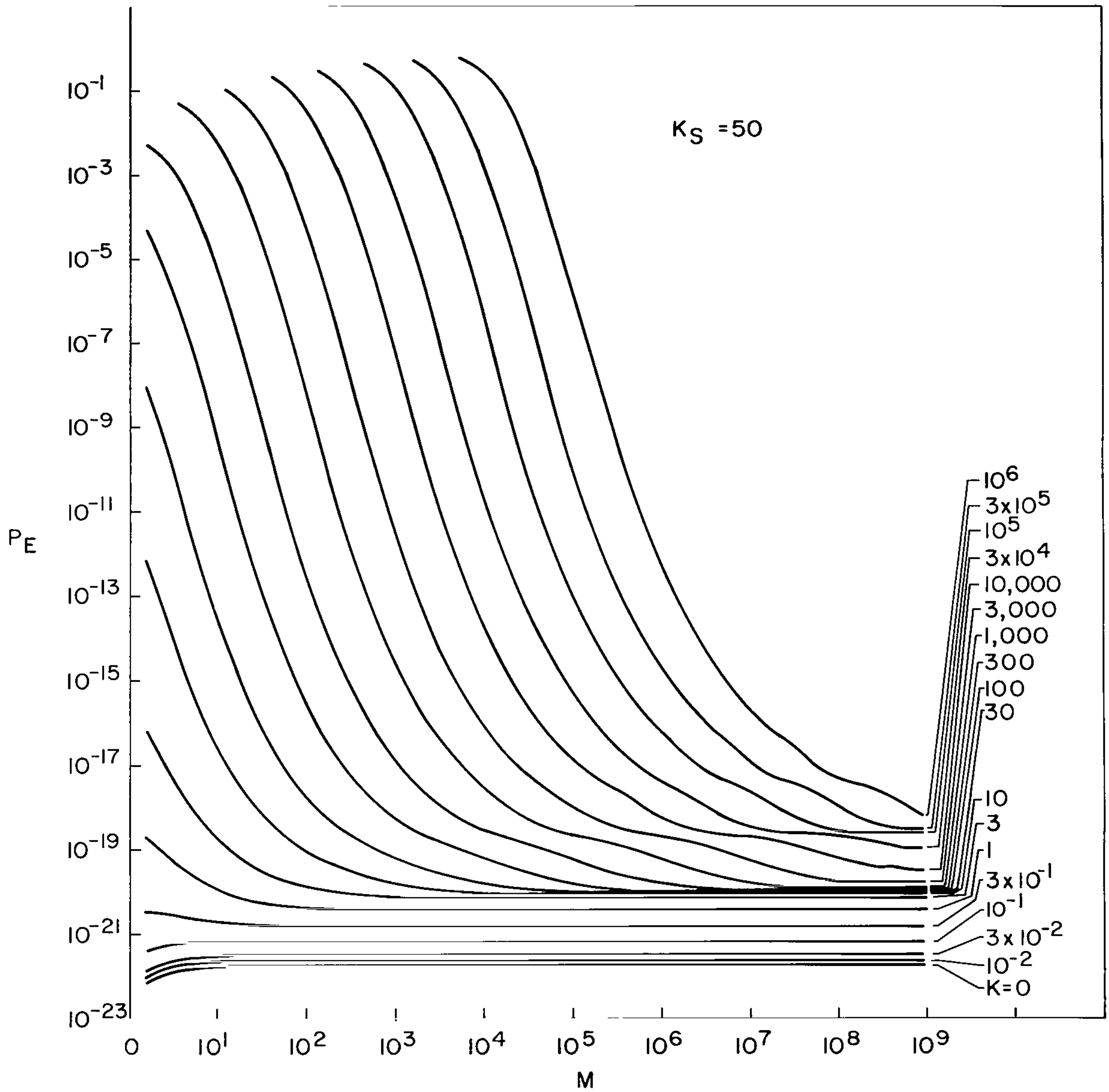

Figure 20.- Error probabilities for $\overline{n}_n T = K$ fixed as a function of M

COMPUTATIONS

The evaluation of P_D and P_E as functions of the parameters K_s, K_n, and M, was accomplished by a set of computer programs which are presented below.

In one case, K_s, K_n, and M are input to the program. In the second case K_n is computed as $K_n = K/M$ where K_s, K and M are input to the program.

The programs were written in Fortran IV and run on the CDC 6400 Computer System at the Smithsonian Astrophysical Observatory in Cambridge, Massachusetts.

DESCRIPTION OF THE FLOW CHART

The constants including tape definitions are set in data statements (Figures 21a, b, and c):

Box 1: Input data and switch values. Set index J to 1. This index counts the number of K_n values.

Box 2: Set index I to 1. This index counts the number of M values. Print headings for tables to be outputted.

Box 3: Set the M value for this loop. In one version of the program K_n is fixed. In the other version it is computed as a function of M.

Box 4: Test magnitude of K_n to prevent overflow or error.

Box 5: Establish initial values of computed parameters.

Box 6: Test magnitude of $(1+B)^M$ to prevent overflow or error.

Box 7: Compute value of term 3.

Box 8: Test for K_n near zero.

Box 9: Compute initial values for PI, Sum PI, Term 2.

Box 10: Compute initial values for term, total.

Box 11: Set index IX to 2. This counts number of times through major computation loop.

Box 12: Compute PX.

Box 13: Test for K_n near zero.

Box 14: Compute PI, Sum PI, Term 2.

Box 15: Compute new B.

Box 16: Test magnitude of $(1 + B)^M$.

Box 17: Compute Term 3.

Box 18: Compute new term and add this total.

Box 19: Test for completion of major loop. If the difference between two successive values of term is sufficiently small, and the index has exceeded a predetermined cutoff value, the loop is completed.

Box 20: Increment IX by 1 .

Box 21: Test for sufficient number of times through the loop.

Box 22: P_D is set to final value of total. P_E is $1-P_D$.

Box 23: Print out table of M, P_D, P_E and their logarithms.

Box 24: Increment index I by 1 .

Box 25: Test for maximum value of M.

Box 26: Increment index J by 1.

Box 27: If J has not exceeded J max, recycle for next value of K (or K_n) .

Box 28: Plot results.

Box 29: Test whether any more data sets are to be computed.

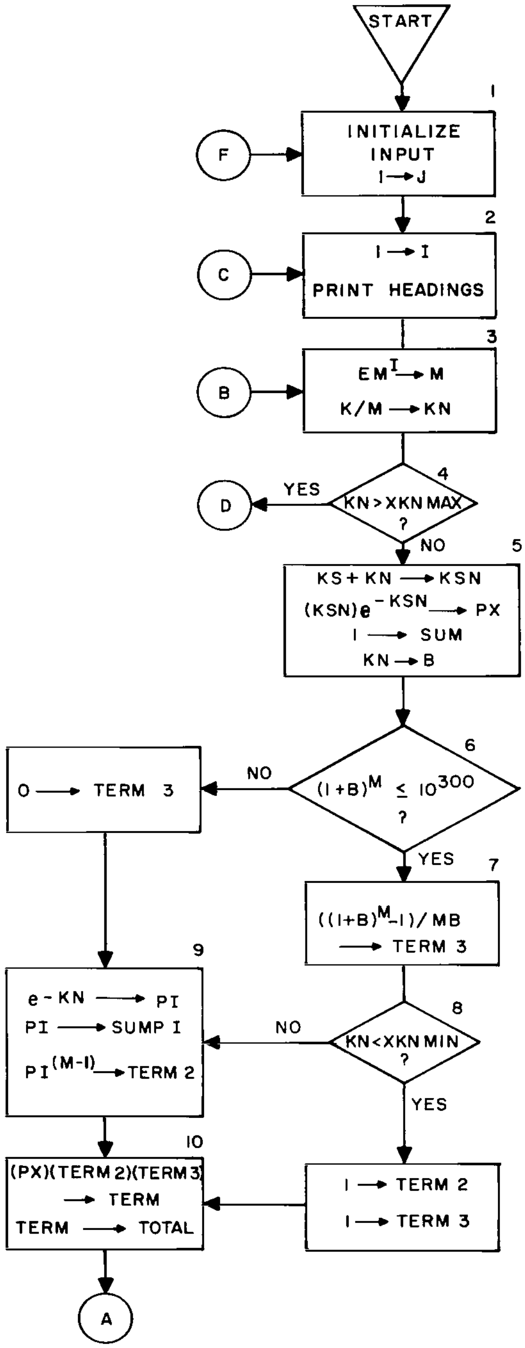

Figure 21a.- Program flow chart

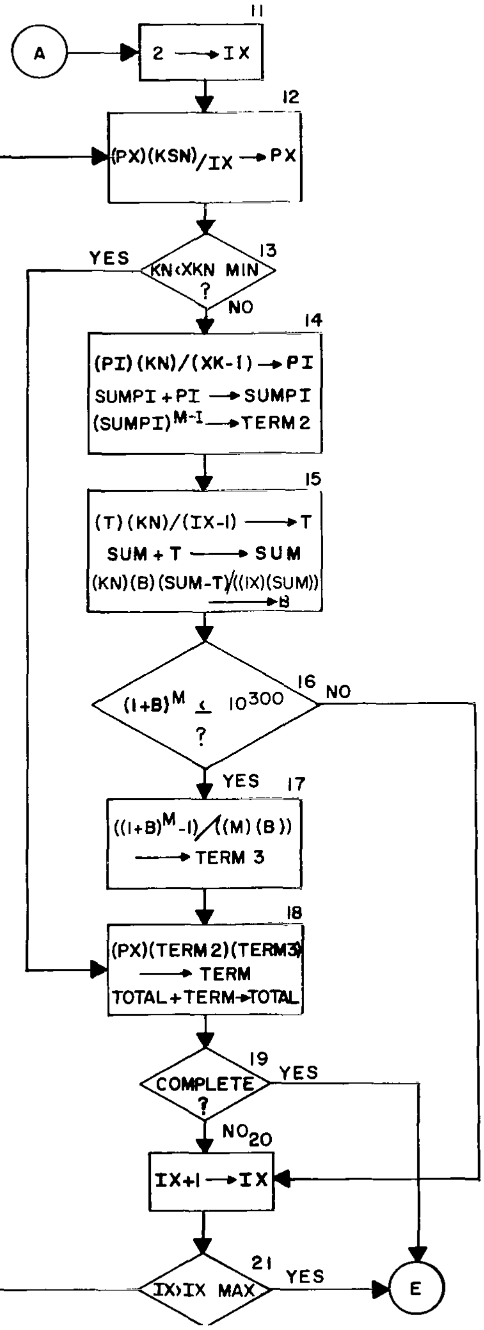

Figure 21b.- Program flow chart

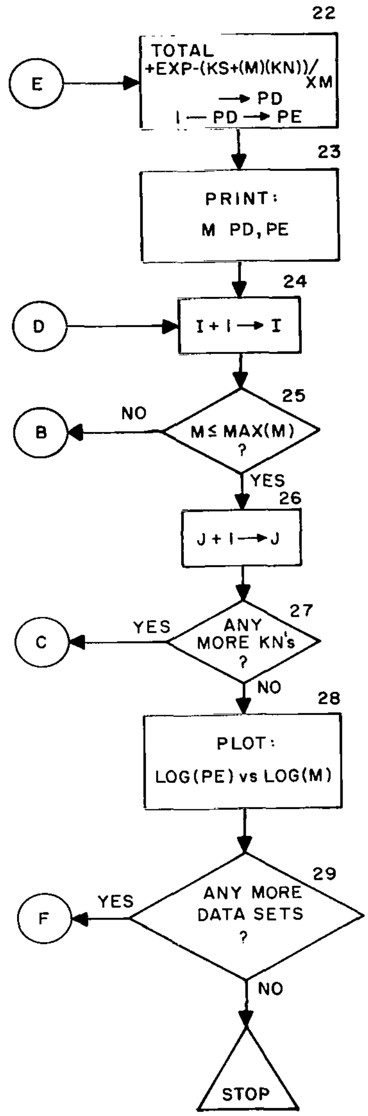

Figure 21c.- Program flow chart

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